New Integrating Circuit and Electrical Analog for Transient Diffusion and Flow*

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A circuit for the electrical simulation of transient diffusion and flow of material between two volumes is described. It allows such processes to be simulated even when forward and reverse transfer rates between the two volumes are unequal. When the reverse rate is zero, the circuit functions as a very-high-input-impedance integrator of wide dynamic range and high accuracy. The circuit will integrate the current through a given resistor, even when this resistor is time varying or nonlinear. For voltage integration, it can have a minimum input impedance approaching 10^9 ohms.

For many purposes, it is desirable to have available a circuit which will accurately integrate a current or voltage without appreciable effect on the quantity to be integrated. The circuit to be described has a wide dynamic range, very high input impedance, and will integrate the current through a resistor which may be time-varying or nonlinear. It was developed as an electrical analogue for flow or diffusion of mass in a biological system between two volumes with unequal forward and reverse transfer rates and will be initially described in terms of this application.

The differential equations governing mass transfer between two volumes v_a and v_b are,

$$\left.\begin{aligned}\frac{dm_a}{dt}&=v_a\frac{dc_a}{dt}=-k_{ab}m_a+k_{ba}m_b=-\rho_{ab}c_a+\rho_{ba}c_b\\ \frac{dm_b}{dt}&=v_b\frac{dc_b}{dt}=+k_{ab}m_a-k_{ba}m_b=\rho_{ab}c_a-\rho_{ba}c_b\end{aligned}\right\}, \quad (1)$$

where the m's are masses, the c's concentrations, k_{ab} the transfer rate constant from volume a to b, k_{ba} the reverse rate constant, and $\rho_{ab}=k_{ab}v_a$, $\rho_{ba}=k_{ba}v_b$. The ρ's are flow rates measured in volume per unit time. Using capital letters to denote parameters in the electrical system we may write the following equations as the electrical analog of (1)

$$\left.\begin{aligned}\frac{dQ_a}{dT}&=C_a\frac{dV_a}{dT}=-\frac{V_a}{R_a}+\frac{V_b}{R_b}=-i_1+i_2\\ \frac{dQ_b}{dT}&=C_b\frac{dV_b}{dT}=\frac{V_a}{R_a}-\frac{V_b}{R_b}=i_1-i_2\end{aligned}\right\}. \quad (2)$$

The solutions of these equations for $V_a(0)\neq 0$ and $V_b(0)=0$ are readily found to be

$$\left.\begin{aligned}V_a(T)&=V_a(0)\times\frac{\left\{\frac{R_aC_a}{R_bC_b}+\exp\left[-\left(\frac{1}{R_aC_a}+\frac{1}{R_bC_b}\right)T\right]\right\}}{[(R_aC_a/R_bC_b)+1]}\\ V_b(T)&=\frac{C_a}{C_b}V_a(0)\times\frac{\left\{1-\exp\left[-\left(\frac{1}{R_aC_a}+\frac{1}{R_bC_b}\right)T\right]\right\}}{[(R_aC_a/R_bC_b)+1]}\end{aligned}\right\}. \quad (3)$$

Since R_a^{-1} determines the forward transfer of charge from C_a to C_b and R_b^{-1} the reverse transfer, we see that when $R_b^{-1}=0$ the charge initially on C_a is finally completely transferred to C_b. The time variations in this case reduce to

$$\left.\begin{aligned}V_a(T)&=V_a(0)e^{-T/R_aC_a}\\ V_b(T)&=\frac{C_a}{C_b}V_a(0)[1-e^{-T/R_aC_a}]\end{aligned}\right\}. \quad (4)$$

It follows from these results, or from Eq. (2), that

$$V_b(T)=\frac{1}{R_aC_b}\int_0^T V_a(\tau)d\tau. \quad (5)$$

Thus, in this case, $V_b(T)$ is, apart from a scale factor, the integral of $V_a(T)$. Since $V_a(T)=i_1(T)R_a$, $V_b(T)$ is

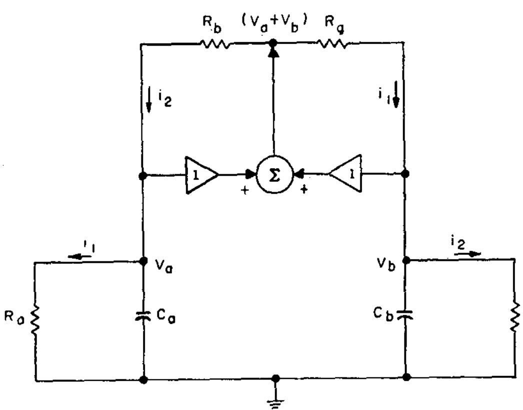

Fig. 1. Block diagram of circuit for simulation of transient diffusion and flow with unequal forward and reverse transfer ratios.

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also proportional to the integral of $i_1(T)$, the current through R_a.

In the forward transfer case where $R_b^{-1}=0$, it is not necessary to restrict the situation to integration of the capacitative discharge current flowing from C_a through R_a to ground. If $C_a=0$ and $i_1\equiv V_a/R_a$ arises from any source of current such as an ion chamber, this current may still be integrated and may be integrated even should R_a be time varying or nonlinear. When $R_b^{-1}=0$, the second of Eqs. (2) reduces to

$$C_b\frac{dV_b}{dt}=\frac{V_a(T)}{R_a(T)}\equiv i_1(T), \qquad (6)$$

whose solution is

$$V_b(T)=\frac{1}{C_b}\int_0^T\frac{V_a(\tau)}{R_a(\tau)}d\tau=\frac{1}{C_b}\int_0^T i_1(\tau)d\tau, \qquad (7)$$

where R_a has here been written as $R_a(\tau)$ to indicate possible dependence of R_a on time or current through it.

A relatively simple circuit realization of Eq. (2) is shown in block diagram in Fig. 1. It makes use of two high-input-impedance unity-gain voltage followers and a nonphase-reversing adder. The two R_a resistors shown are ganged together as are the two R_b resistors. It may be readily verified that this circuit realizes Eqs. (2) and that, for example, with $R_b^{-1}=0$, any initial charge on C_a is transferred to C_b. This block diagram has already been mentioned in conjunction with a description of an electrical analog computer for simulating tracer diffusion in biological systems.[1,2] In Fig. 2 we show how the heart of the circuit, the followers and adder, have been realized using operational amplifiers.

Where it is desired to integrate a voltage rather than a current and to accomplish such integration with exceptionally high input impedance, then R_b^{-1} is taken zero as before, and both C_a and the left R_a of Fig. 1 are eliminated. If the voltage to be integrated, V_a, is applied at the (high-input-impedance) input to the left-hand voltage follower, the current i_1, through the remaining R_a resistor is still V_a/R_a and Eq. (7) again applies with R_a a constant resistor. Thus, the voltage V_b will be proportional to the integral of the input voltage. Proper operation of the circuit of Fig. 1 depends on several factors. First, the minimum input impedance of the followers must be as high as possible. Second, the two R_a resistors must be made as equal as practical, as must the R_b resistors. Finally, the voltage-transfer ratio from either follower input to the adder output should be very close to unity. If it is greater than unity, the circuit is regenerative, and the total amount of charge in the system is not time-independent but will increase. In order to make the resistors as equal as possible, they have been built as ganged decades, each

[1] J. R. Macdonald, Proc. Natl. Simulation Conf. 1, 29.1 (1956).

[2] Macdonald, Perry, Madison, and Seldin, Radiation Research 6, 585 (1957).

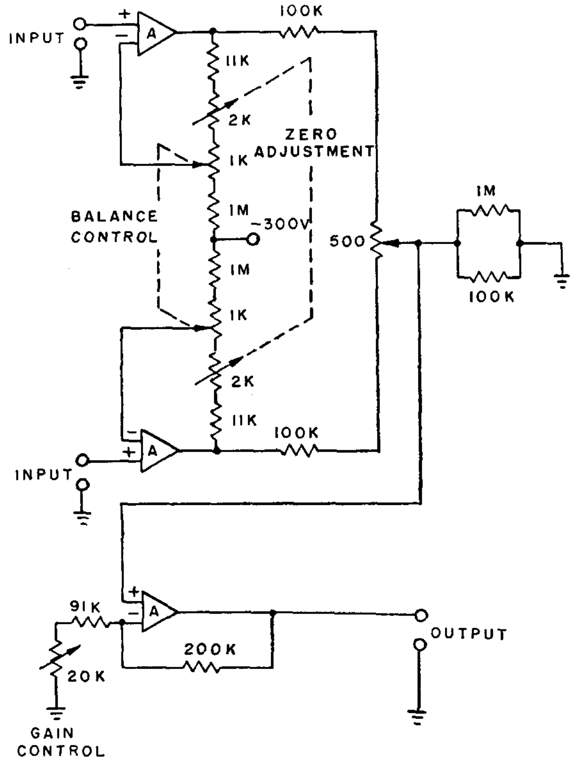

FIG. 2. Circuit of high-input-impedance, nonphase-reversing adder. The amplifiers are Philbrick Model K2-W operational units.

pair covering the range from 9.999 to 0.001 megohms. Corresponding resistors of a pair were selected equal to a small fraction of a percent.

In the adder circuit of Fig. 2, the operational amplifiers are Philbrick Model K2-W's with an open loop voltage gain of 15 000, minimum input impedance approaching 10^3 megohms, dynamic range of ± 50 volts, and rise time of 2 μsec. Adding takes place across the 100K and 1M resistors in parallel. The loss in adding is made up by the final amplifier which has a feedback voltage gain of approximately three. Since it is desired that a voltage applied at either input appear at the output with no change of dc level, the inherent dc offsets of the operational amplifiers are cancelled by means of the zero adjustment. Exact equality between the dc offsets of the two followers is achieved by means of the balance control. The 500-ohm potentiometer is used to achieve exact equality between the two closely matched 100K resistors at the follower outputs. Finally, the gain control allows adjustment of the input-output voltage transfer ratio to exactly unity. By cancelling the dc offset of the final amplifier at the followers, interaction between the zero adjustment and the gain control is greatly reduced compared to offset cancellation at the final amplifier itself. After final adjustment, an input of say +10 or +20 volts applied to either input with the other at ground or grounded through 100K produced an output of the same magnitude to considerably better than 0.01% measured with a Fluke Model 801 digital voltmeter. With non zero voltages applied to both inputs, their sum appeared at the output with the same high precision and linearity. From the amplifier specifications, one would not expect a zero drift appreciably greater than 15 to 20 mv/day at the output of the adder, and the measured drift was found to be of this order of magnitude after warm up.

Various operational tests of the circuit of Figs. 1 and 2 have been made. One of the most stringent is to start with $R_b^{-1}=0$, $R_a^{-1}\neq 0$, and $V_a(0)\neq 0$. The charge initially on C_a is then transferred to C_b. If R_a^{-1} is then set to zero and R_b^{-1} is made greater than zero, the charge on C_b is transferred back to C_a. It was found possible to transfer such an initial charge back and forth 10 to 100 times in a period of a few minutes without any appreciable loss of charge during the process. Mylar-dielectric capacitors were used for C_a and C_b in this experiment. In monitoring this operation, another operational amplifier with very high input impedance was used to observe the voltage V_a or V_b without drawing appreciable current. It is worth pointing out that if the voltage transfer ratio of the adder circuit of Fig. 2 is made very slightly greater than unity so that the circuit without loading would be slightly regenerative, the very small loading of the voltage follower circuits can be effectively cancelled so neither regeneration nor overall loss of charge occurs. This condition can be maintained to a high degree of accuracy for many minutes.

The circuit of Figs. 1 and 2 was built for the biological analog simulator computer already mentioned. Its operation in this instrument also affords a good test of its adequacy as an electrical analogue of diffusion or flow with unequal forward and reverse transfer rates. The computer allows the transient output of the circuit to be plotted to appreciably better than one percent during charge transfer from C_a to C_b or vice versa. To this degree of accuracy, Eq. (3) checked perfectly for a wide variety of C_a/C_b and R_a/R_b values. In particular, with $R_a=R_b$ so that forward and reverse transfer rates were equal, behavior indistinguishable from that of a single resistor R_a joining C_a and C_b was observed.

Finally, it should be mentioned that the use of the circuit as an integrator with $R_b^{-1}=0$ and R_a time-varying or nonlinear is only possible when both R_a resistors of Fig. 1 vary with time or current in exactly the same way, since only in this case will the current i_1, in the left-hand R_a be the same as that in the right-hand R_a.