Rayleigh-Wave Dissipation Functions in Low-Loss Media

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Summary

A simple formula is derived relating the specific dissipation factors for Rayleigh, compressional, and shear waves in relatively low-loss materials. Calculations and a table are presented which allow an unknown specific dissipation factor to be obtained directly from the two known factors for the other types of waves. Results apply for any realizable ratio of any two of the three elastic phase velocities.

1. Introduction

The surface Rayleigh waves which travel in a solid half-space in addition to body compressional and shear waves are of particular importance for earthquake detection at large distances, where they predominate over three-dimensional body waves. As Press & Healy (1957) have pointed out, it is of interest to relate the absorption coefficients of Rayleigh waves to those of shear and compressional waves for materials showing energy dissipation and dispersion. These authors have derived a useful expression for this purpose for low-loss media which we shall here simplify, slightly generalize, and reduce to numerical applicability.

It will prove most convenient to deal with the specific dissipation factor, $1/Q$, for each type of wave, where $2\pi/Q$ is the ratio of energy dissipated per stress cycle to the peak energy stored (Knopoff & MacDonald 1958). The quantity $1/Q$ is a useful quantity for comparison with experiment because it is independent of the detailed mechanism of energy dissipation.

2. Dispersion and absorption

Whenever dissipation occurs, the resulting dispersion makes it necessary to distinguish between three kinds of wave velocities: the elastic velocities in the absence of dissipation, the corresponding phase velocities with dissipation, and the complex velocities. Using the subscripts A, B, and C to denote quantities relating to compressional, shear, and Rayleigh waves, the elastic velocities may be obtained from

$$v_A = [(\lambda+2\mu)/\rho]^{\frac{1}{2}}, \tag{1}$$

$$v_B = [\mu/\rho]^{\frac{1}{2}}, \tag{2}$$

$$[2-(v_C/v_B)^2]^4 = 16[1-(v_C/v_A)^2][1-(v_C/v_B)^2], \tag{3}$$

where ρ is the density and λ and μ are Lamé's moduli.

A damped harmonic wave travelling in the x direction involves the factor $\exp[i\omega t-\gamma x]$, where $\gamma\equiv\alpha+i\beta$ and α and β are the attenuation and phase factors respectively of the wave. For a given type of wave, the phase velocity V is given by ω/β while the complex velocity is $V' = \omega/(-i\gamma) \equiv \omega/k$. Further, when dispersion is present, the real equation (3) is replaced by the corresponding equation involving the complex velocities V_i'. Following Press & Healy, this complex equation can be used to determine a relation between $1/Q_A$, $1/Q_B$, and $1/Q_C$ in low-loss materials. It is assumed in the following that these three quantities refer to waves all having the same period.

From their definitions, it can be shown that

$$V_i/V_i' = 1-i(\alpha_i/\beta_i), \tag{4}$$

$$1/Q_i = 2(\alpha_i/\beta_i)/[1-(\alpha_i/\beta_i)^2]. \tag{5}$$

Now if the dissipation is relatively low so that $1/Q\leqslant 0{\cdot}1$, (5) can be expanded and simplified to yield, with negligible error,

$$V_i/V_i' \cong 1-i(1/2Q_i), \tag{6}$$

where terms in $(1/2Q_i)^2$ have been neglected. To the same order of approximation, it can be shown that for either the Voigt dissipation model considered by Press & Healy or for loss arising from creep (Lomnitz 1957) $V_A\cong v_A$ and $V_B\cong v_B$. Taking $V_C\cong v_C$ as well and defining

$$a\equiv(v_C/v_A)^2,\ b\equiv(v_C/v_B)^2, \tag{7}$$

the complex-velocity Rayleigh equation becomes

$$\left[2-b\left\{\frac{1-i(1/2Q_B)}{1-i(1/2Q_C)}\right\}^2\right]^4 = 16\left[1-b\left\{\frac{1-i(1/2Q_B)}{1-i(1/2Q_C)}\right\}^2\right]\left[1-a\left\{\frac{1-i(1/2Q_A)}{1-i(1/2Q_C)}\right\}^2\right]. \tag{8}$$

When (8) is expanded retaining terms up to $1/Q_i$ only, one obtains an equation from the real part identical with (3) and an equation from the imaginary part involving the $1/Q_i$'s. By using (3) to simplify the latter, or by taking imaginary parts of the logarithms of (8), one finally finds

$$\frac{1}{Q_C} = (1-m)\frac{1}{Q_B}+m\frac{1}{Q_A}, \tag{9}$$

where

$$m = \frac{a(2-b)(1-b)}{a(2-b)(1-b)-b(1-a)(2-3b)}. \tag{10}$$

Although it is obtained more generally, Equation (9) is a simplified but equivalent form of the equation obtained by Press & Healy (1957).

The result (9) is a simple linear equation connecting the $1/Q_i$'s and is valid to high approximation when none of them exceed $0{\cdot}1$. This equation can be used to obtain any unknown $1/Q_i$ given the other two and knowledge of m. It is interesting to note that $1/Q$ is a more sensitive measure of loss than is dispersion of the velocities which, to the present approximation, is negligible.

In order to make (9) useful, the quantity m, which is a complicated function of v_A, v_B, and v_C, has been calculated, using a digital computer, as a function of v_B/v_A and is presented in Table 1 and Figure 1. Table 1 also gives v_C/v_B as a function of v_B/v_A, and the resulting curve is shown in Figure 1 with its ordinate on the right. Now, if any two of the elastic phase velocities are known, the third can be obtained using the Figure or the Table. Then, after the corresponding value of m is obtained, the unknown $1/Q_i$ can be obtained using (9) and known values of the other $1/Q_i$'s.

Table 1

Numerical results

v_B/v_A	v_C/v_B	m
0	0·95531	0
0·05	0·95516	$3·2411 \times 10^{-4}$
0·10	0·95469	$1·3233 \times 10^{-3}$
0·15	0·95389	$3·0823 \times 10^{-3}$
0·20	0·95271	$5·7570 \times 10^{-3}$
0·25	0·95112	$9·6002 \times 10^{-3}$
0·30	0·94903	$1·5008 \times 10^{-2}$
0·35	0·94633	$2·2597 \times 10^{-2}$
0·40	0·94286	$3·3347 \times 10^{-2}$
0·45	0·93837	$4·8851 \times 10^{-2}$
0·50	0·93253	$7·1789 \times 10^{-2}$
0·55	0·92476	0·10685
0·60	0·91419	0·16257
0·65	0·89937	0·25512
0·70	0·87785	0·41586
0·75	0·84549	0·70455

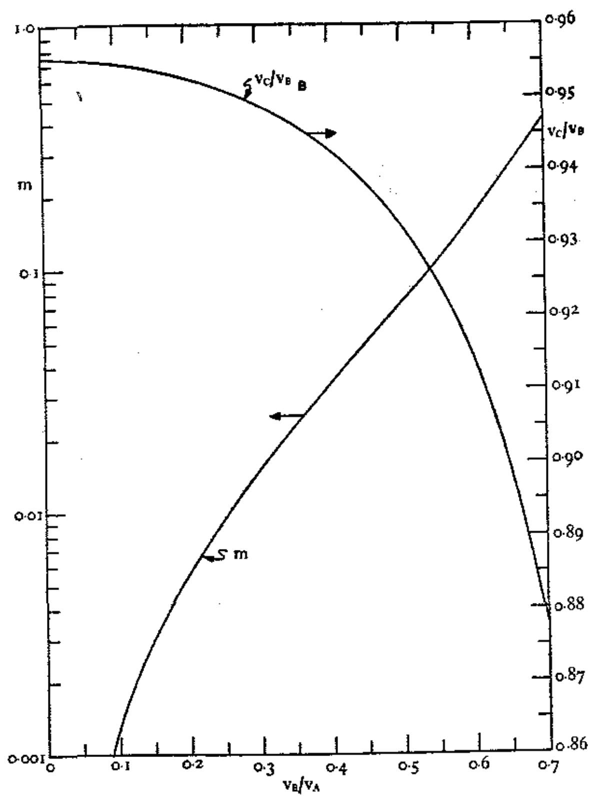

FIG. 1. The quantities m (log scale) and v_C/v_B versus v_B/v_A.

It will be noted that the abscissa extends only from 0 to 0·7. The ratio v_B/v_A may be expressed for isotropic material as

$$(v_B/v_A) = [(1-2\nu)/2(1-\nu)]^{\frac{1}{2}}, \qquad (11)$$

where ν is Poisson's ratio. For an incompressible body $\nu = \frac{1}{2}$ and $v_A = \infty$. The minimum value of ν is zero; thus the maximum value of v_B/v_A is $2^{-\frac{1}{2}}$. The quantity m increases continuously beyond this physical limit, however, and reaches 9·1 for $v_B/v_A = 0·95$ and infinity for $v_B/v_A = 1$.

When one or more of the $1/Q_i$'s are appreciably greater than 0·1, the exact complex equivalent of (3) must be solved using (4) and (5) in their exact forms. Solution of the resulting cubic equation with complex coefficients will be difficult but is usually unnecessary since the condition $1/Q_i \leqslant 0·1$ is usually well met in most situations of physical interest.

Finally, it should be pointed out that for $v_B/v_A \geqslant 0·567008$ there are three real roots of (3) rather than the single one which appears for smaller v_B/v_A. As Jeffreys (1952) has shown, however, it is only the smallest of the three roots which is of physical significance for Rayleigh wave transmission.

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