and

$$Q_n = \frac{Q_d}{2\pi}\int_0^{2\pi}(1-\alpha\cos\theta)^{1-\nu}\cos n\theta\,d\theta,\ n\geq 0. \quad (10)$$

Comparison of (2) and (8) as well as (6) and (10) shows that if

$$C_n/C_d = f(\alpha, n, \nu) \quad (11a)$$

then

$$Q_n/Q_d = f(\alpha, n, \nu - 1). \quad (11b)$$

General Values of ν

With

$$k^2 \equiv \frac{2\alpha}{1+\alpha}. \quad (12)$$

some changes in the integration variables allow (6) to be written

$$C_n = \frac{2C_d(-1)^n}{(1+\alpha)^\nu\pi}\int_0^{2\pi}\frac{\cos 2n\beta}{(1-k^2\sin^2\beta)^\nu}\,d\beta. \quad (13)$$

Note that $1>k>0$ for $1>\alpha>0$. Expansion of the denominator in (13) yields known integrals,[2] and there results, after some manipulation,

$$C_n = \frac{C_d}{(1+\alpha)^\nu}\sum_{r=n}^{\infty}P(r,n)T(\nu,r)k^{2r},\ n\geq 0, \quad (14)$$

with

$$P(r,n) = \frac{[\Gamma(r+1)]^2}{\Gamma(r+1-n)\Gamma(r+1+n)}, \quad (15)$$

and

$$T(\nu,r) = \frac{\Gamma(\nu+r)}{\Gamma(\nu)}\ \frac{\Gamma(r+1/2)}{\Gamma(1/2)}\ \frac{1}{[\Gamma(r+1)]^2}. \quad (16)$$

where $\Gamma(z)$ is the gamma function. Note that the summation in (14) starts with the $r=n$ term since all prior terms are zero. Also, $C_n>0$ for $n\geq 0$.

From (11) it is evident that

$$Q_n = Q_d(1+\alpha)^{1-\nu}\sum_{r=n}^{\infty}P(r,n)T(\nu-1,r)k^{2r},$$
$$n\geq 0. \quad (17)$$

Here with Q_d negative, $Q_0<0$, and $Q_n>0$ for $n\geq 1$.

Direct integration of (6) in terms of known functions is possible; with

$$\alpha = \frac{2t}{1+t^2}, \quad (18)$$

there results

$$C_n = (1+t^2)^\nu t^n\frac{\Gamma(n+\nu)}{\Gamma(\nu)\Gamma(n+1)}$$
$$\cdot{}_2F_1(\nu, n+\nu; n+1; t^2)C_d,\quad n\geq 0, \quad (19)$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function.[3] Note $1>t>0$ for $1>\alpha>0$. Since[3]

$${}_2F_1(\nu, n+\nu; n+1; x)$$
$$= \frac{\Gamma(n+1)\Gamma(\nu)\Gamma(1-\nu)}{(-1)^n\Gamma(\nu+n)\Gamma(1-\nu+n)}\ \frac{1}{(1-x)^{\nu-1}}\ \frac{d^n}{dx^n}$$
$$\cdot[(1-x)^{\nu+n-1}{}_2F_1(\nu,\nu;1;x)]. \quad (20)$$

[2] W. Magnus and F. Oberhettinger, "Formulas and Theorems for the Special Functions of Mathematical Physics," Chelsea Publishing Co., New York, N. Y., p. 5; 1949.

[3] *Ibid.*, p. 11.

C_n can be obtained from C_0 with $x=t^2$ in (20). Also, the use of

$$(1+t)^{2\nu}{}_2F_1(\nu,\nu;1;t^2) = {}_2F_1(\nu, 1/2; 1; k^2)\ ^{4} \quad (21)$$

in (19) and (20) establishes the equivalence of C_n given by (19) and (14). The expression for C_n in terms of the hypergeometric function has, aside from its intrinsic interest, the advantage that various transformations, recursion formulas, and identities are available. For example, with k^2 near 1, (14) is slowly convergent, but the equivalence of (14) and (19) enables a highly convergent series in $1-k^2$ to be obtained easily.[4] Also from (19) and (20) and other hypergeometric function relationships, it can be shown that

$$C_{n+2} = -\frac{(n+\nu)}{(n-\nu+2)}C_n$$
$$+\frac{(1+t^2)}{t}\ \frac{(n+1)}{(n-\nu+2)}C_{n+1}. \quad (22)$$

The hypergeometric function involved in C_0 can be expressed as a fractional-order Legendre function; thus[5]

$${}_2F_1\left(\nu,\nu;1;\tanh^2\frac{u}{2}\right)$$
$$= \left(\cosh^2\frac{u}{2}\right)^\nu \mathcal{P}_{-\nu}(\cosh u), \quad (23)$$

where $t=\tanh u/2$, but no useful tables of these functions seem to be available.

From (11) and (19) it is clear that

$$Q_n = (1+t^2)^{\nu-1}t^n\frac{\Gamma(n+\nu-1)}{\Gamma(\nu-1)\Gamma(n+1)}$$
$$\cdot{}_2F_1(\nu-1, n+\nu-1; n+1; t^2)Q_d,$$
$$n\geq 0, \quad (24)$$

and that the relationships noted in (20)–(23) relating to C_n apply equally well to Q_n with the substitution indicated in (11).

The Case of $\nu=\frac{1}{2}$

In this important special case, evaluation results in terms of the tabulated complete elliptic integrals from (21)[6] or from (13).[7] Thus,

$$C_0 = \frac{2C_d}{(1+\alpha)^{1/2}\pi}K, \quad (25)$$

$$C_1 = C_0 - \frac{4C_d}{(1+\alpha)^{1/2}\pi k^2}[E-(1-k^2)K], \quad (26)$$

etc. K and E are the complete elliptic integrals of the first and second kind, respectively, with modulus k.

Similarly[8]

$$Q_0 = \frac{2Q_d}{\pi}(1+\alpha)^{1/2}E, \quad (27)$$

$$Q_1 = -Q_0 - \frac{4Q_d}{3\pi k^2}(1+\alpha)^{1/2}$$
$$\cdot[(1-2k^2)E + K(k^2-1)], \quad (28)$$

etc.

The transmission phase shift in a parametric amplifier can be dependent on C_0. The influence of the amplitude of the pump

[4] *Ibid.*, p. 9.

[5] *Ibid.*, p. 56.

[6] *Ibid.*, p. 10.

[7] P. F. Byrd and M. F. Friedman, "Handbook of Elliptic Integrals for Engineers and Physicists," Springer-Verlag, Berlin, Germany, pp. 192–193; 1954.

[8] *Ibid.*, p. 194. The integrals involved here can be obtained by differentiating with respect to k.

voltage, V_0, on this parameter as derived from (25) is given by

$$\frac{\partial C_0}{\partial V_0} = -\frac{C_0}{2V_0}\left[1-\frac{1}{1-\alpha}\ \frac{E}{K}\right]. \quad (29)$$

After completion of the above analysis and preparation of this note, it was brought to our attention that related work had already appeared in the document literature.[9] However, appearance of the independent work reported here still seems worthwhile. It not only calls more widespread and deserved attention to the prior work but also points out the unified and generalized results which accrue from the fact, apparently not previously noted, that the capacitance and charge coefficients can be expressed in terms of the hypergeometric function.

S. Sensiper
R. D. Weglein
Res. Labs.
Hughes Aircraft Co.
Culver City, Calif.

[9] D. I. Breitzer, R. Gardner, J. C. Greene, P. P. Lombardo, B. Salzberg, E. W. Sard, and K. Siegel, "Quarterly Progress Reports, Application of Semiconductor Diodes to Low-Noise Amplifiers, Harmonic Generators, and Fast-Acting TR Switches," Airborne Instruments Lab., Huntington, Long Island, N. Y., Rept. 4589-M-1, Appendix IV, September, 1958; Rept. 4589-I-4, Appendix E, June 1959; Contract AF 30(602)-1854.

Space-Charge Capacitors for Parametric Amplifiers*

In parametric amplifiers, which depend upon the voltage-sensitive nonlinearity of a reactive element for their action, it is desirable that such nonlinearity be as large as possible over the voltage range of interest. In addition, low-noise and high-frequency response require minimum in-phase (resistive) losses in the reactive element.[1] In view of the great current interest in parametric amplifiers, a short summary of some of the work on space-charge capacitors pertinent to their use as varactors is presented below.

Some time ago, the author investigated in detail the capacitance behavior of certain semiconductor space-charge regions near blocking electrodes[2–6] and pointed out[6] that such systems should be useful for parametric

* Received by the IRE, January 29, 1960.

[1] W. E. Danielson, "Low noise in solid state parametric amplifiers at microwave frequencies," *J. Appl. Phys.*, vol. 30, pp. 8–15; January, 1959.

[2] J. R. Macdonald and M. K. Brachman, "Exact solution of the Debye-Huckel equations for a polarized electrode," *J. Chem. Phys.*, vol. 22, pp. 1314–1316; August, 1954.

[3] J. R. Macdonald, "Static space-charge effects in the diffuse double layer," *J. Chem. Phys.*, vol. 22, pp. 1317–1322; August, 1954.

[4] J. R. Macdonald, "Theory of the differential capacitance of the double layer in unadsorbed electrolytes," *J. Chem. Phys.*, vol. 22, pp. 1857–1866; November, 1954.

[5] J. R. Macdonald, "Static space charge and capacitance for a single blocking electrode," *J. Chem. Phys.*, vol. 29, pp. 1346–1358; December, 1958.

[6] J. R. Macdonald, "Static space charge and capacitance for two blocking electrodes," *J. Chem. Phys.*, vol. 30, pp. 806–816; March, 1959.

amplifiers since they may yield exponential dependence of capacitance on applied voltage as well as low resistance losses. In the earlier work,[2-4] the situation was considered where mobile charge of both signs is blocked (no transfer of charge carriers between the charge-containing material and the metallic electrodes) at one or two electrodes, and an essentially exponentially increasing dependence of differential capacitance on applied bias voltage was obtained. Here, charges of both signs are mobile, and the capacitance increases for either sign of the applied voltage because for either sign an accumulation layer of excess charge (space-charge) forms in the region of the blocking electrode(s). This situation applies to the case of intrinsic semiconduction in a solid or to a completely dissociated univalent electrolyte.

Later, the case was considered where charge of only one sign is mobile; the charge carriers are blocked at one[5] or two[6] electrodes; and mobile charge can recombine to any degree with fixed charge of opposite sign arising from the dissociation or ionization of neutral centers distributed uniformly throughout the material. This situation corresponds to photoconduction in insulators (*e.g.*, F-centered alkali-halide crystals) or to extrinsic semiconduction in materials sufficiently doped or with a large enough bandgap that the effects of mobile minority charge carriers may be neglected. In the case of complete dissociation, where the recombination parameter R is zero, the situation is essentially that treated by Schottky[7] and Spenke[8] in their theory of rectification. This theory was applied to such systems as copper-oxide and selenium rectifiers which show partial (noninfinite barrier height) blocking at a metallic electrode.

The capacitance behavior of a system with only negative charges mobile, but blocked at one electrode,[5] is shown in Fig. 1 for different degrees of recombination R. Here C_d is the differential capacitance, and C_0 is its value in the limit of zero potential difference across the space-charge region. For a positive potential difference across the space-charge region, essentially exponentially increasing capacitance is obtained no matter what the value of R since a negative space-charge accumulation layer is set up at the blocking electrode. On the other hand, a negative potential difference yields a charge depletion layer for $R=0$ whose capacitance eventually decreases as the inverse square root of the bias voltage. For large R, however, recombination essentially mobilizes the immobile charge, and again an exponentially increasing capacitance can occur for a limited voltage range.

Because it is not easy and is sometimes impossible to produce an electrode-solid interface which approximates well to ideal blocking behavior over an appreciable applied voltage range, the effect of an artificial blocking layer was also considered in detail in the earlier work.[5,6] Here, the charge-containing material is abutted by an essentially charge-free region between it and a metallic electrode. Such a region can be produced by an oxide layer on the material or by the interposition of a thin insulating layer of such material as mica or mylar. If this charge-free region is sufficiently thin, its capacitance, which is essentially in series with any space-charge capacitance, may be much larger initially than the latter, and a considerable range of exponential over-all capacitance increase may still be possible. Fig. 2 shows the calculated behavior for different values of the ratio C_r/C_{ser}, where C_{ser} is the capacitance of the charge-free layer and C_c is the combined capacitance of C_d and C_{ser}. Note especially that the value of C_r/C_{ser} determines the potential range over which the effect of the space-charge capacitance is important. Because the potential across the charge-free region and that across the space-charge region are inversely proportional to their respective capacitances, and the space-charge capacitance is a nonlinear function of the potential across it, a transcendental equation must be solved to obtain the final dependence of C_c on over-all applied potential.[9] The solution of such an equation can be avoided, however, if the potential across the space-charge layer is taken as the main variable and the over-all applied potential derived from it. This situation has not been well

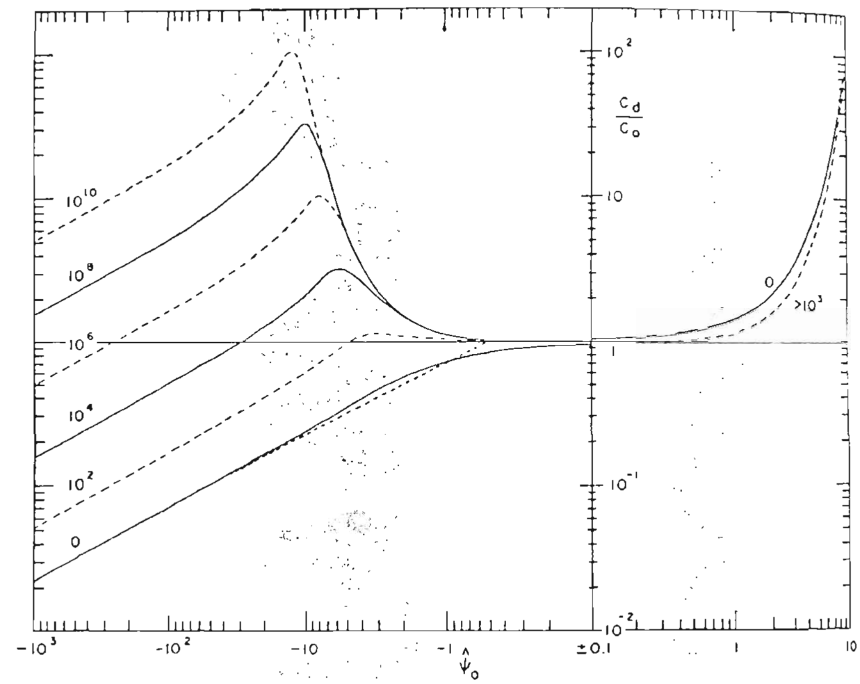

Fig. 1—Relative space-charge differential capacitance versus normalized potential ($\hat{\psi}_0 = e\psi_0/kT$) across the space-charge region for various values of the recombination parameter R.

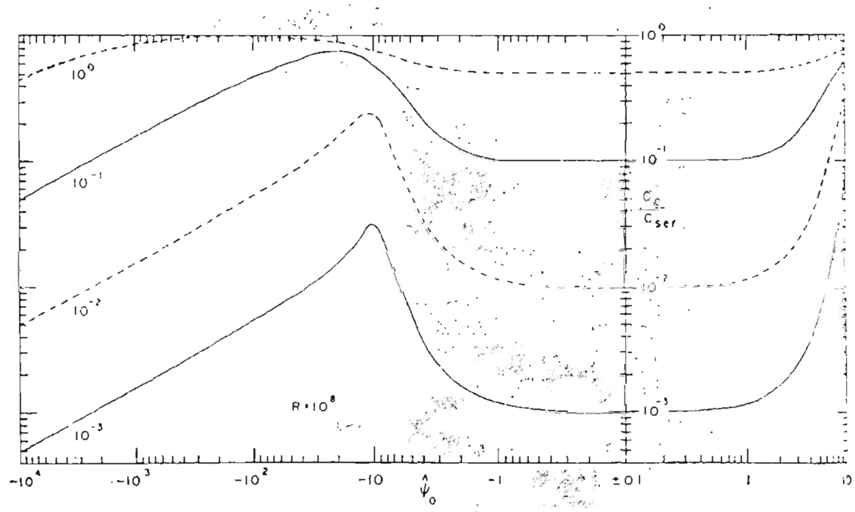

Fig. 2—Dependence on normalized potential of relative space-charge differential capacitance of a system comprising a voltage-independent capacitance C_{ser} in series with a voltage-dependent space-charge capacitance C_d for various values of the ratio C_0/C_{ser} and $R = 10^8$.

[7] W. Schottky, "Vereinfachte und erweiterte Theorie der Randschichtgleichrichter," *Zeits. für Phys.*, vol. 118, pp. 539-592; September-October, 1941.

[8] E. Spenke, "Zur Randschichttheorie der Trockengleichrichter," *Zeits. für Phys.*, vol. 126, pp. 67-83; January-February, 1949.

[9] On using the condition that the charge in the initial layer is equal to that in the space-charge region (*i.e.*, no discontinuity in the normal component of dielectric displacement at the interface between the regions), one sees that the static (not differential) capacitance of the space-charge region must be used in this equation. Note that the potential variable in Figs. 1 and 2 includes in it the contribution of any "built-in" or barrier potential.

appreciated in the past, but was discussed as early as 1954 in the case of charges of both signs mobile.[4]

A capacitor made up of a plate of n-type silicon with an oxide layer covered by a metallic contact on one side and either the same situation or an ohmic contact on the other side may possibly be a good realization of the state of affairs discussed above. Such a capacitor has been under investigation in a number of laboratories for some time. Because an oxide or other intermediate layer may be made essentially charge-free with consequent very low conduction, the resistive losses of the over-all device may, in principle, be considerably lower than those of a p-n diode, especially in regions where exponential capacitance variation may be obtained. Terman[10] has, in fact, found a change in over-all capacitance as high as 10:1 for a two-volt bias change. This change seems to be largely an exhaustion decrease rather than an accumulation increase in capacitance, but Dewald[11] has apparently observed such an increase for zinc oxide in contact with an electrolyte, a blocking-electrode situation. Wallmark[12] has suggested the use of an oxide layer device to yield improved gate control of a field effect transistor. Recently, Pfann and Garrett have also discussed the properties of such systems in a qualitative way and have pointed out some of the frequency-response characteristics of these space-charge devices.[13]

J. Ross Macdonald
Central Res. Labs.
Texas Instruments
Dallas, Texas

[10] L. Terman, "Silicon Oxide Capacitors," Solid-State Electronics Lab., Stanford University, Stanford, Calif., Consolidated Quarterly Status Rept. No. 4, pp. 20–27; July 1 to September 30, 1959. See also J. L. Moll, "Variable capacitance with large capacity change," 1959 IRE WESCON Convention Record, pt. 3, pp. 32–36.
[11] J. F. Dewald, "The charge and potential distributions at the zinc oxide electrode," *Bell Sys. Tech. J.*; to be published.
[12] J. T. Wallmark, U. S. Patent No. 2900531, August 18, 1959.
[13] W. G. Pfann and C. G. B. Garrett, "Semiconductor varactors using surface space-charge layers," Proc. IRE, vol. 47, pp. 2011–2012; November, 1959.

An Electrostatically Focused Electron Beam Parametric Amplifier*

Electron beam parametric amplifiers employing magnetic focusing have been described.[1,2] In one of these devices,[1] gain is achieved by pumping the fast space charge wave. In the other device,[2] gain is achieved by pumping the fast cyclotron wave. Thus, in the latter device, the magnetic field plays a vital role in addition to that of focusing the beam. The electron beam parametric amplifier which is proposed here employs electrostatic focusing of an electron sheet beam and achieves gain by pumping the fast wave of the natural resonant electron frequency associated with the electrostatic focusing fields.

Consider a sheet beam which passes between a series of pairs of identical planar plates at alternate dc voltages V_1 and V_2 (see Fig. 1). The voltage difference between adjacent pairs of plates forms an electrostatic field which acts to focus the beam by creating a time average restoring force, accelerating the electrons towards the plane midway between the plates.[3] The electrons in such a focusing system have a natural transverse resonant frequency f_e first described by Adler.[3] A more exact expression for f_e is[4]

$$f_e = 0.187 \times 10^6 \left(\frac{\epsilon}{a}\right) \sqrt{V_0} \cdot \left[\sum_{n=1,3,5,7,\cdots}^{\infty} \frac{1}{\cosh^2 (n\pi d/2a)} \right]^{1/2}, \quad (1)$$

where

V_0 = the space average beam voltage

$$= \frac{V_1 + V_2}{2},$$

$$\epsilon = \frac{V_2 - V_1}{2V_0}.$$

a = the periodicity of the focusing plates (Fig. 1), and

d = the separation between opposing plate pairs (Fig. 1).

In both (1) and (2), all terms are expressed in mks units.

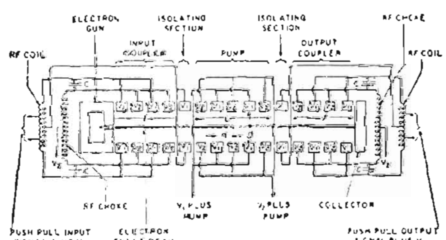

Fig. 1—A simplified drawing of an electrostatically focused parametric amplifier.

The means by which parametric amplification is achieved in this electrostatically focused system can be most simply described by dividing the plates into three groups with respect to RF: the input coupler, the pump, and the output coupler (Fig. 1). The plates of the input coupler are connected so that they form the capacitance of a tank circuit, wherein all of the plates on each side of the beam are at the same RF potential. The tank circuit should be designed to resonate at approximately the frequency of the signal to be amplified, f_s. The output coupler is identical to the input coupler. The leads for the input and output signals are tapped off the RF coils (Fig. 1) to effect a transformer action which reduces the mismatch between the resistive loading of the coupler[5] and the characteristic impedance of the transmission line. Balanced input and output arrangements are shown; however, an unbalanced system could also be used.

The theory of Cuccia applies to the input and output couplers. Thus, if $f_s \approx f_e$, and if the Cuccia equation[5] is satisfied, all of the energy introduced across the plates of the input coupler will be transferred to the sheet beam traveling through the coupler plates in the form of a transverse oscillatory motion of the beam. Simultaneously, the inherent noise will be stripped off the beam. The output coupler performs the inverse function of the input coupler; *i.e.*, it acts to convert the transverse motion of the sheet beam entering the coupler into an RF signal in the load connected to the output coupler.

Gain is achieved by increasing the amplitude of the transverse oscillation of the electron sheet beam as it travels from the input coupler to the output coupler. This increase is accomplished in the pumping section by superimposing a frequency of approximately twice the electron resonant frequency on both voltages V_1 and V_2 in a push-pull manner, as shown in Fig. 1. It may also be possible to achieve satisfactory pumping by modulating only one of these two voltages and keeping the other one constant. The modulation of the focusing voltages varies the restoring force acting on the electron beam. The increase of amplitude of the transverse oscillation of the electron beam is achieved in a manner analogous to the building up of the amplitude of oscillation of a pendulum by varying either the restoring force or the length of the pendulum at exactly twice its natural resonant frequency.[6] In direct comparison with this pendulum model, and assuming the pumping signal is properly synchronized with respect to f_e, the gain will vary as $e^{2\alpha L}$, where αL is given by

$$\alpha L = \frac{\pi \Delta f_e}{u_a} L \quad (2)$$

and where

L = the length of the pump,

Δf_e = the peak variation of f_e due to pumping,

u_a = the average beam velocity.

If $f_s \approx f_e$ but $f_s \neq f_e$, an idler frequency equal to the difference between the pump and signal frequencies will also be present on the beam as it leaves the pumping section. Under this condition,[2] the gain of the signal frequency will vary as $\cosh^2 \alpha L$, and the gain of the idler frequency will vary as $\sinh^2 \alpha L$. The value of Δf_e corresponding to the peak values of the applied pumping voltages may be determined from (1). There is a limit to the amplitude of the pumping signal that may be applied to the electrostatically focused parametric amplifier. The applied pumping voltages must be limited to that range over

* Received by the IRE, January 28, 1960.
[1] A. Ashkin, "Parametric amplification of space charge waves," *J. Appl. Phys.*, vol. 29, pp. 1646–1651; December, 1958.
[2] R. Adler, G. Hrbek, and G. Wade, "The quadrupole amplifier, a low-noise parametric device," Proc. IRE, vol. 47, pp. 1713–1723; October, 1959.
[3] R. Adler, O. M. Kromhout, and P. A. Clavier, "Resonant behavior of electron beams in periodically focused tubes for transverse signal fields," Proc. IRE, vol. 43, pp. 339–341; March, 1955.
[4] W. E. Waters, "Periodic focusing of thin electron sheet beams," *DOFL Tech. Rev.*, vol. 2, pp. 1–25; July, 1959.
[5] C. L. Cuccia, "The electron coupler—a developmental tube for amplitude modulation and power control at ultra-high frequencies," *RCA Rev.*, vol. 10, p. 278 (18); June, 1949.
[6] G. Wade and R. Adler, "A new method for pumping a fast space-charge wave," Proc. IRE, vol. 47, pp. 79–80; January, 1959.