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Energy dissipation and attenuation under high-loss conditions

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Abstract. The conventional relation $\Delta = \pi \tan \delta$ between free-vibration logarithmic decrement Δ and loss tangent or internal friction was shown by Parke to fail for high damping conditions. Here it is shown that the relation fails in the opposite direction to that found by Parke when Δ is a spatial decrement derived from attenuation of a plane wave travelling in a passive medium. The following exact relations are found for this case: $\Delta = 2\pi\alpha/\beta = 2\pi \mid \tan \frac{1}{2}\delta \mid$, $Q^{-1} = 2(\alpha/\beta)/\mid 1 - (\alpha/\beta)^2 \mid = \mid \tan \delta \mid$, where α and β are the attenuation factor and phase factor of the wave in the material and Q is the quality factor of the material. Frequency response curves are shown for $R \equiv Q^{-1}/(\Delta/\pi) \geqslant 1$, Q^{-1} and Δ/π for the cases of both non-resonant absorption of power and resonant absorption. For resonant absorption the phase lag δ may exceed $\pi/2$. The deviation between the present results and those of Parke is shown to arise from intrinsic difficulties in precisely defining and measuring period (free vibration) or wavelength (wave motion) under high-loss conditions.

1. Introduction

Although it is well known that the usual simple equations relating dissipation factor and such quantities as attenuation of a plane wave or logarithmic decrement break down for high-loss conditions, the exact relations which hold for any possible loss are not well known. Here we shall investigate these relations for the specific case of internal friction losses in solids. The results will apply as well, however, to electrical losses in dielectrics, semiconductors, or metals.

There are a number of methods of measuring the internal friction energy-loss parameters of anelastic linear materials (Kolsky 1953, Zener 1960, Klima 1962). By measuring the heat produced during cyclic deformation one can determine the ratio of energy dissipated per cycle throughout the specimen ΔW to the maximum stored deformation energy W. Measurement of the decay of the amplitude of oscillations of a freely vibrating body yields the logarithmic decrement Δ, a well-defined quantity provided that the decay is exponential and that the damping is not so large that the motion is aperiodic. By exciting forced oscillations of a material with a harmonic force of constant amplitude and variable frequency, the compliance and resonance curve of the material may be obtained and related to internal friction in the material. Finally, measurement of the attenuation of the amplitude of an elastic wave in the medium may be employed to yield the attenuation constant α. Note that these various methods may yield loss parameters related to different elastic processes such as shear, volume compression, longitudinal deformation, etc.

For small damping, it is conventional to write

$$\Delta = \pi \tan \delta \tag{1}$$

where $\tan \delta$ is the internal friction dissipation factor or loss tangent, and Δ and $\tan \delta$ should properly both be obtained from measurements involving the same type of deformation. Here δ is the phase angle by which a sinusoidal strain lags an applied sinusoidal stress. It is frequently termed the loss angle. Recently, Parke (1966)† has shown that for a standard linear solid with appreciable damping, $\Delta > \pi \tan \delta$ in the frequency region near the maxima of Δ and $\tan \delta$. For large damping he finds that the usual relation fails by a significant

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† Parke denotes the logarithmic decrement by Q rather than Δ.

ratio. Note that Δ and $\tan\delta$ are only connected by equation (1) for low-loss conditions when the measurements which led to them were properly carried out at the same frequency. While it is relatively easy to change the frequency in $\tan\delta$ measurements, it is not as easy to do so in free vibration experiments.

The present work leads to the surprising result that for a general model of the solid (which includes the standard linear solid model), $\Delta < \pi \tan\delta$ when Δ is derived not from free vibrations but from attenuation of a longitudinal plane pressure wave travelling in the medium.

2. Analysis

Consider first the application of an alternating longitudinal stress $\sigma = \sigma_0 \exp(i\omega t)$ to a general linear viscoelastic material. For sufficiently small σ_0, such that ΔW and W are proportional to the square of the amplitude of vibration (and thus the material remains linear), the response will be an alternating longitudinal strain $\epsilon = D(i\omega)\,\sigma$, where $D(i\omega)$ is the complex longitudinal compliance. This notation and much of that to follow conforms with the recommendations of the Committee on Nomenclature of the Society of Rheology (Leaderman 1957). Let $D(i\infty) \equiv D_{\mathrm{g}}$ and $D(i0) \equiv D_{\mathrm{e}}$ and consider, for simplicity, a material with infinite viscosity. Then

$$\begin{aligned} D(i\omega) &\equiv D_{\mathrm{g}} + D_{\mathrm{a}}(i\omega) \\ &\equiv D_{\mathrm{g}}(1 + Q(i\omega)) \end{aligned} \tag{2}$$

where $D_{\mathrm{a}}(i\omega)$ is the part of the compliance arising from anelastic effects,

$$Q(i\omega) \equiv \frac{D_{\mathrm{a}}(i\omega)}{D_{\mathrm{g}}} \equiv \frac{qD_{\mathrm{a}}(i\omega)}{D_{\mathrm{d}}} \tag{3}$$

and

$$q \equiv \frac{D_{\mathrm{d}}}{D_{\mathrm{g}}} \equiv \frac{D_{\mathrm{e}} - D_{\mathrm{g}}}{D_{\mathrm{g}}}. \tag{4}$$

In these equations, $D_{\mathrm{d}} \equiv D_{\mathrm{a}}(i0)$, D_{e} is the equilibrium compliance, and D_{g} is the instantaneous or elastic compliance. In terms of the unrelaxed modulus $M_{\mathrm{U}} \equiv D_{\mathrm{g}}^{-1}$ and the relaxed modulus $M_{\mathrm{R}} \equiv D_{\mathrm{e}}^{-1}$ the quantity q equals $(M_{\mathrm{U}} - M_{\mathrm{R}})/M_{\mathrm{R}}$, termed the relaxation strength by Zener (1960). In the dielectric case $q \equiv (\epsilon_{\mathrm{s}} - \epsilon_\infty)/\epsilon_\infty$, where ϵ_{s} is the 'static' dielectric constant and ϵ_∞ the 'infinite' frequency dielectric constant. Clearly, either the electrical or mechanical q may appreciably exceed unity for some materials. We may now write

$$Q(i\omega) \equiv J(\omega) - iH(\omega) \tag{5}$$

$$J(\omega) = (q/D_{\mathrm{d}})\,\mathscr{R}(D_{\mathrm{a}}(i\omega)) \tag{6}$$

$$H(\omega) = -(q/D_{\mathrm{d}})\,\mathscr{I}(D_{\mathrm{a}}(i\omega)) \tag{7}$$

and observe that $Q(0) = J(0) = q$. The quantity $Q(i\omega)$ is a normalized complex compliance accounting in a completely general way for the anelastic part of the behaviour only. Its relations to other relaxation or retardation parameters are discussed elsewhere (Macdonald and Brachman 1956, Macdonald and Barlow 1963)†.

From the definitions, it now follows that

$$D(i\omega) = |D(i\omega)| \exp(-i\delta) \tag{8}$$

$$\sin\delta = \frac{H(\omega)}{\{(1 + J(\omega))^2 + (H(\omega))^2\}^{1/2}} \tag{9}$$

$$\cos\delta = \frac{1 + J(\omega)}{\{(1 + J(\omega))^2 + (H(\omega))^2\}^{1/2}} \tag{10}$$

$$\tan\delta = \frac{H(\omega)}{1 + J(\omega)}. \tag{11}$$

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† In both these references $J(0)$ is normalized such that $J(0) = 1$.

The quantity $\sin \delta$ is known as the power factor. The above relations hold exactly for all physically possible values of the phase angle δ. This angle may sometimes be negative, indicating a phase lead instead of lag. For example, in an active system which supplies rather than dissipates energy $H(\omega)$ may be negative over part of the ω range. Further, as we shall see later, $1 + J(\omega)$ and hence $\cos \delta$ may sometimes become negative even in a passive system.

It has been shown elsewhere (Gross 1948, Macdonald 1961) that the parameter $Q^{-1} \equiv \Delta W/2\pi W$ is equal to $\tan \delta$ as given by (11). Here, the quality factor Q should not be confused with the normalized compliance $Q(i\omega)$. When $\sin \delta$ or $\cos \delta$ may be separately negative, the connection between Q^{-1} and $\tan \delta$ must be slightly modified. It is reasonable that Q be negative for an active system, but Q should not be negative when only $\cos \delta$ is negative, since the system remains passive. We shall therefore write

$$Q^{-1} \equiv |\tan \delta| \operatorname{sgn}(H(\omega))$$

and ignore the sgn factor hereafter since we shall not deal with active systems herein.

The displacement $U(x, t)$ for a longitudinal plane wave of infinitesimal amplitude incident on a homogeneous material may be written as

$$U(x, t) = U_0 \exp\{-(\alpha + i\beta)x\} \exp(i\omega t) \tag{12}$$

for travel in the $+x$ direction. Here $\alpha \equiv \alpha(\omega)$ is the attenuation factor and $\beta \equiv \beta(\omega) = \omega/V$ is the phase factor. The quantity V is the phase velocity, equal to the elastic velocity V_e in the absence of attenuation and power loss. Let us therefore define $\beta_0 \equiv \omega/V_e$. For comparison, in the free vibration case considered by Parke (1966), the strain, or angle of twist in torsional oscillations, is proportional to $\exp\{-(\xi + i\omega_0)t\}$. Here ξ is the temporal damping constant and ω_0 is the radial frequency of oscillation.

In previous work (Macdonald 1961) it has been shown that

$$\alpha + i\beta = i\beta_0\{(1 + J(\omega)) - iH(\omega)\}^{1/2}.$$

This equation leads to

$$\alpha = \pm \beta_0\{(1 + J(\omega))^2 + (H(\omega))^2\}^{1/4} |\sin \tfrac{1}{2}\delta| \tag{13}$$

$$\beta = \pm \beta_0\{(1 + J(\omega))^2 + (H(\omega))^2\}^{1/4} |\cos \tfrac{1}{2}\delta| \tag{14}$$

where the $+$ sign is appropriate for a wave travelling in the $+x$ direction in a passive material. From (13) and (14) we may derive the exact passive-system relations

$$\alpha/\beta \equiv |\tan \tfrac{1}{2}\delta| \tag{15}$$

and

$$Q^{-1} \equiv |\tan \delta| = \frac{2\alpha/\beta}{|1 - (\alpha/\beta)^2|}. \tag{16}$$

The general correction term in (16) to the usual approximate relation (Knopoff 1965) $Q^{-1} \simeq 2\alpha/\beta$ can also be derived for a less general case from equations given by Marvin, Aldrich and Sack (1954).

Let the wavelength λ of the plane wave be written as $\lambda = 2\pi V/\omega = 2\pi/\beta$. Then the wave is reduced in amplitude on travelling one wavelength by the factor $\exp(-\alpha\lambda) \equiv \exp(-\Delta)$ and thus

$$\Delta = 2\pi\alpha/\beta = 2\pi |\tan \tfrac{1}{2}\delta| \tag{17}$$

an exact expression for Δ no matter what its magnitude. It is interesting to note the differences between equations (17) and (1). If we define

$$R \equiv \frac{Q^{-1}}{\Delta/\pi} = \left|\frac{\tan \delta}{2\tan \frac{1}{2}\delta}\right| = |1 - \tan^2 \tfrac{1}{2}\delta|^{-1} = \tfrac{1}{2}|\sec \delta + 1| \tag{18}$$

then it is clear that $R > 1$ for $\delta^2 > 0$ in the present case, whereas Parke found $R < 1$ in his somewhat different situation. In both cases, however, equation (1) is well satisfied provided δ is sufficiently small. When $(\frac{1}{2}\delta)^2 \ll 1$, we have from (18) $R \simeq 1 + (\frac{1}{2}\delta)^2$ and $Q^{-1} = |\delta|$, where δ is in radians. In the opposite extreme as $\delta \to \pi/2$, $\alpha \to \beta$, $\Delta \to 2\pi$ and $Q \to 0$. Note that the $Q = 0$ amplitude ratio is $\exp(-2\pi) \simeq 1/535$. When $Q = 1$, on the other hand, $\Delta = 2\pi(\sqrt{2} - 1) \simeq 2{\cdot}607$ and the corresponding amplitude ratio is about $1/13{\cdot}6$.

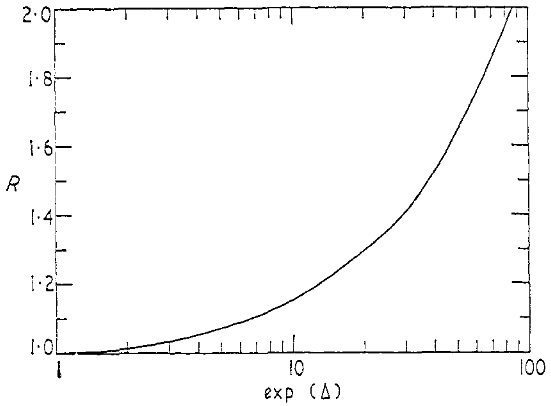

Figure 1. The ratio $R \equiv Q^{-1}/(\Delta/\pi)$ plotted against the inverse damping amplitude ratio, exp (Δ).

In figure 1 we have plotted R against the inverse amplitude ratio, exp (Δ). The $Q = 1$ value of R is still only about 1·2, showing that very low Q values are required for R to be much greater than unity. Parke found that when $q = 1$ and $\exp(\Delta) = 3{\cdot}78$, the minimum $R \simeq 1/1{\cdot}2$. In the present case this value of $\exp(\Delta)$ corresponds to only $R \simeq 1{\cdot}044$, a considerably smaller effect.

The complex admittance $Q(i\omega)$ may describe a system with one or many discrete relaxation times or with a continuous distribution of such times. Elsewhere (Macdonald 1963), we have suggested a general distribution of relaxation times function which seems particularly appropriate for many viscoelastic materials. Here it is of interest to investigate the behaviour of R with frequency for two different types of compliance each involving, for simplicity, a single relaxation time.

Let τ_0 be the effective relaxation time of the material and define the normalized frequency variable $W \equiv \omega\tau_0$. The simplest non-resonant compliance function is of the Debye type and leads to

$$J_{\mathrm{D}} = q/(1 + W^2) \tag{19}$$

$$H_{\mathrm{D}} = qW/(1 + W^2). \tag{20}$$

These equations are formally equivalent to those associated with the standard linear solid (Zener 1960). Note that even though the material is non-resonant, equations (11), (19) and (20) lead to a resonance curve for tan δ.

Alternatively, we shall consider a Lorentzian compliance function which itself exhibits resonance. Such a function leads to (Macdonald and Brachman 1956)

$$J_{\mathrm{L}} = \frac{q}{2}\left\{\frac{1 + W_0(W + W_0)}{1 + (W + W_0)^2} + \frac{1 - W_0(W - W_0)}{1 + (W - W_0)^2}\right\} \tag{21}$$

$$H_{\mathrm{L}} = \frac{qW}{2}\left[\{1 + (W + W_0)^2\}^{-1} + \{1 + (W - W_0)^2\}^{-1}\right] \tag{22}$$

where $W_0 \equiv \omega_0\tau_0$ is the normalized resonant frequency. Note that when $W_0 = 0$, (21) and (22) reduce to (19) and (20), respectively. Such reduction does not occur with the more usual one-term Lorentz dispersion formulae for resonant absorption. The present equations are thus more appropriate and general (Fröhlich 1958).

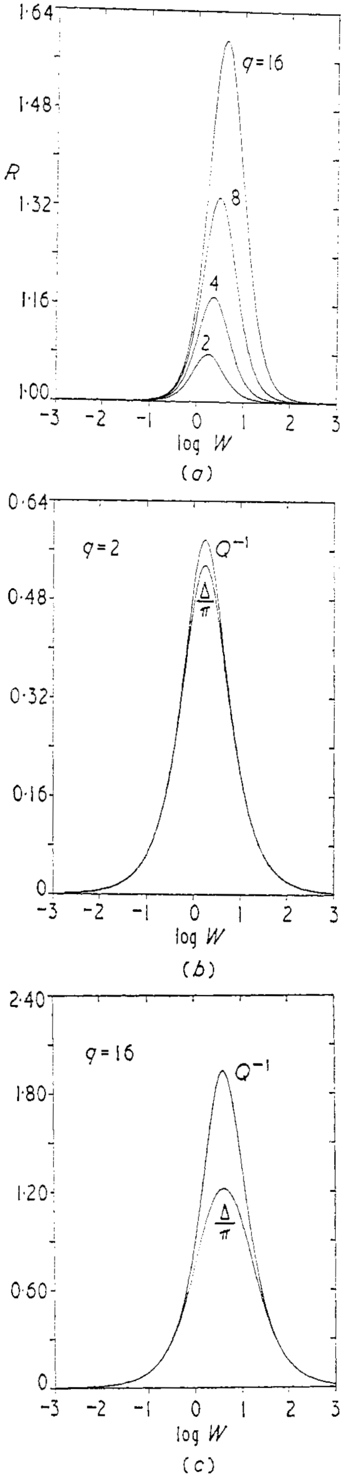

Figure 2. The quantities R, Q^{-1} and Δ/π plotted against $W \equiv \omega\tau_0$ for various q values and $W_0 = 0$.

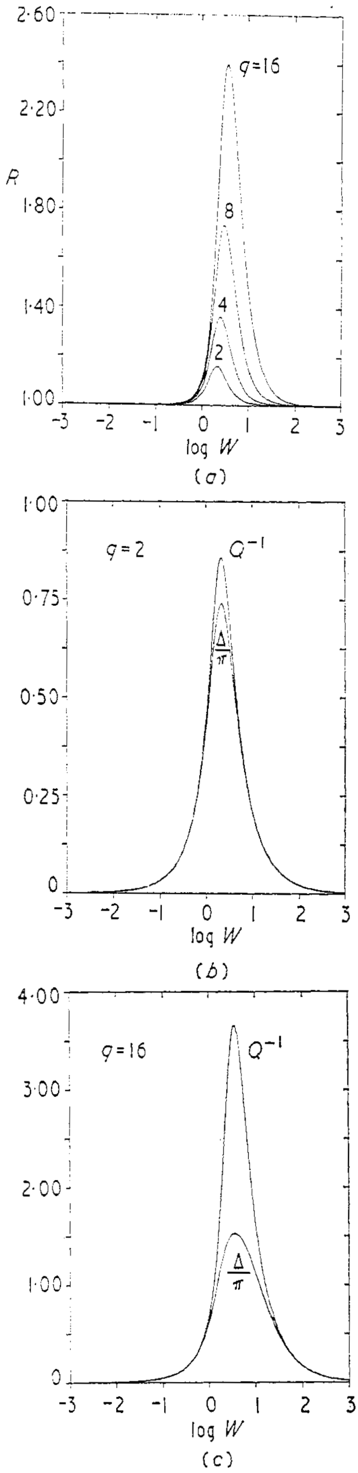

Figure 3. The quantities R, Q^{-1} and Δ/π plotted against $W \equiv \omega\tau_0$ for various q values and $W_0 = 1$.

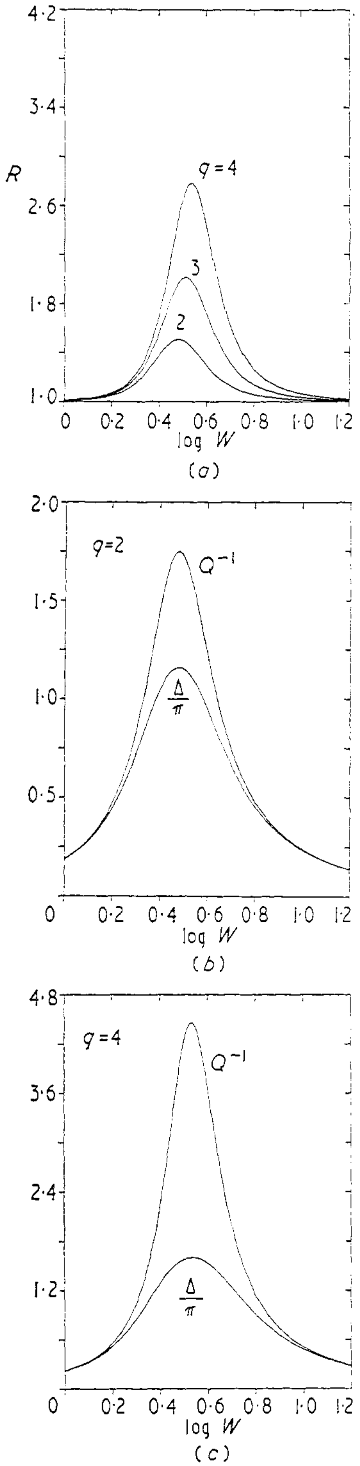

Figure 4. The quantities R, Q^{-1} and Δ/π plotted against $W \equiv \omega\tau_0$ for various q values and $W_0 = 2$.

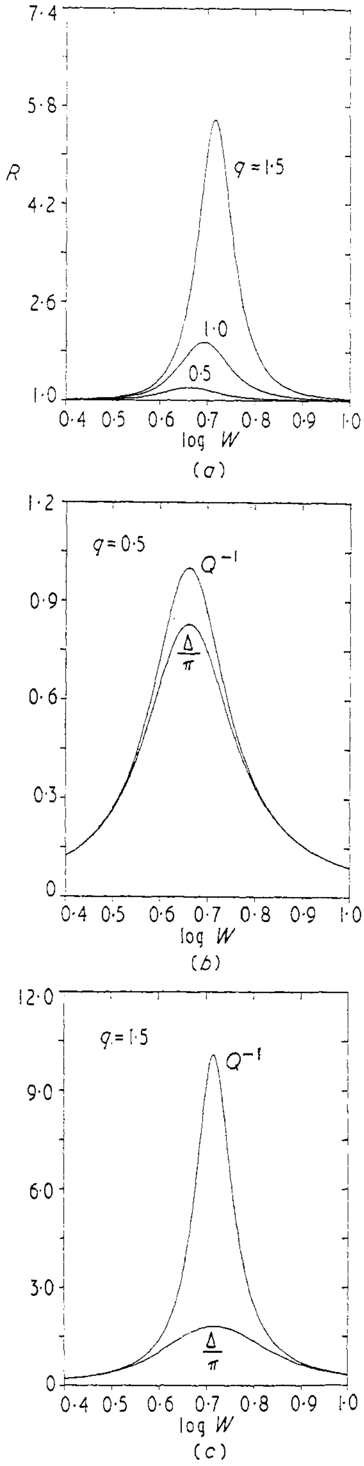

Figure 5. The quantities R, Q^{-1} and Δ/π plotted against $W \equiv \omega\tau_0$ for various q values and $W_0 = 4$.

3. Frequency-response results

Figures 2–5 show frequency response curves for R, Q^{-1} and Δ/π for a variety of q values and $W_0 = 0$, 1, 2 and 4, respectively. The curves were computer calculated and were automatically plotted. The change in the vertical scales as q and W_0 increase should be especially noted as well as the narrowing of the curves as W_0 increases. In addition to the increases in curve heights shown as q and W_0 increase, it is clear that the frequency corresponding to the peak of the curves also increases with them and does not occur at $\omega = \tau_0{}^{-1}$ ($\log W = 0$) but at greater frequencies.

Let W_{M} be the normalized frequency at which Q^{-1} reaches its maximum. In the $W_0 = 0$ case, $W_{\mathrm{M}} \equiv (1 + q)^{1/2} \equiv (D_{\mathrm{e}}/D_{\mathrm{g}})^{1/2}$. When $W = W_{\mathrm{M}}$,

$$Q^{-1} = q/2(1 + q)^{1/2} = (D_{\mathrm{e}} - D_{\mathrm{g}})/2(D_{\mathrm{e}}D_{\mathrm{g}})^{1/2}.$$

For $W_0 = 0$, infinite q is thus required to reduce Q to zero. It is of interest to note that when $q \ll 1$, $Q^{-1} \simeq \frac{1}{2} q = H_{\mathrm{D}}$ at $W = 1$.

When $W_0 > 0$, W_{M} is given by a considerably more complicated expression. Further, note that for the same q all $W_0 > 0$ curves lie above the corresponding $W_0 = 0$ curves. As W_0 increases and ω_0 exceeds $\tau_0{}^{-1}$ by a larger and larger factor ($W_0 > 1$), it turns out, in fact, that J_{L} can go negative and finally even exceed unity in magnitude. The larger W_0, the smaller the value of q required for $1 + J(\omega)$ to become negative at large W values. When $W^2 \gg W_0{}^2 \gg 1$, it is readily found that $J_{\mathrm{L}} = -1$ when $q \simeq (W/W_0)^2$. For simplicity, we have limited the W_0 and q combinations used in figures 2–5 to those which keep $J(\omega) > -1$ and thus restrict δ to the range $0° \leqslant \delta < 90°$.

At low frequencies values of Q^{-1} as high as 20 have been reported (Benbow and Wood 1963) for organic glasses which show viscosity. Although viscosity contributions have been omitted from the compliance functions used herein, they may readily be added to them in a simple way.

4. Discussion

Although it may seem surprising that R should exceed unity for the present situation of power absorption from a plane stress wave and be less than unity when Δ is derived from a free vibration experiment, such a difference can and must arise from the differences in the two experimental situations, which lead to different results for Δ even when both Δ's are defined in conventional and reasonable ways.

When Δ increases to unity or beyond, it becomes increasingly difficult to define precisely and measure. In the plane wave situation, the wavelength in the material must be known to obtain Δ, but the error in establishing a definite λ characteristic of the system grows as Δ increases. Similarly, to obtain Δ in the free vibration case one must know the period, but this too becomes more difficult to establish precisely as Δ increases. Although these problems may be reduced, but not entirely avoided, by replicating measurements and using statistical analysis, these techniques will still not save one from the situation that at sufficiently large Δ the imprecision in Δ will approach Δ itself in magnitude.

Although the frequency associated with a wave of the type of equation (12) may be very well defined, the wavelength, or wave number k, is actually distributed and not of the form $\beta\delta(k - \beta)$, where $\delta(k - \beta)$ is a Dirac δ-function. If we write $f(x) \equiv \exp\{-(\alpha + i\beta)\, x\}$, then Fourier transformation of $f(x)$ yields its mate

$$g(k) = \{\alpha + i(\beta - k)\}^{-1}. \tag{23}$$

The quantity $|\, g(k) \,|$ exhibits resonance character and shows, as expected, that the most probable wave number is $k = \beta$. The $\delta k \equiv |\, \beta - k_{\mathrm{e}} \,|$ which produces $|\, g(k_{\mathrm{e}})/g(\beta) \,| = \mathrm{e}^{-1}$ is $\delta k = \alpha(\mathrm{e}^2 - 1)^{1/2} \simeq 2{\cdot}5\alpha$. If we note that $\delta x \simeq \alpha^{-1}$, then $\delta k \delta x \simeq 2{\cdot}5$, not in conflict with the Heisenberg uncertainty relation $\delta k \delta x \gtrsim 1$ (Mott and Sneddon 1948).

In the temporal damping case, the characteristic function is of the form $\exp\{-(\xi + i\omega_0)t\}$, where ω_0 is the most probable radial frequency of vibration. Fourier analysis then leads analogously to $\delta\omega \simeq 2{\cdot}5\xi$, again consonant with the uncertainty relation $\delta\omega\delta t \gtrsim 1$ if δt is taken as ξ^{-1}.

To first order in Δk, we may write $|\delta\Delta/\Delta| \simeq |\delta k/k|$. Suppose that Δ can be determined sufficiently accurately by replication and statistics that its most probable value is found to be $\Delta = 2\pi\alpha/\beta$, involving β, the most probable value of k. We may then write

$$|\delta\Delta/\Delta| \simeq 2{\cdot}5\alpha/\beta = 2{\cdot}5\Delta/2\pi \simeq \Delta/2{\cdot}5$$

which holds when $|\delta\Delta| \ll \Delta$. This condition in turn requires $\Delta \ll 2{\cdot}5$. (The same conclusion may be reached in an analogous way for temporal damping.) Clearly, when $\Delta \gtrsim 1$, therefore, the distribution in k produced by large α makes it impossible to attribute a very precise meaning to Δ itself. Further, for such large Δ conditions, if two different experiments, each with a distributed or imprecise Δ, produce Δ values whose difference appreciably exceeds $2\Delta^2/2{\cdot}5$ in magnitude, then something is wrong with one or both of the experiments or definitions of Δ. On the other hand, if the different Δ's should be closer together than the above Δ difference, no such difficulty can be asserted *a priori*.

Parke (1966) found for $q = 1$ that $\Delta_{\max} \simeq 1{\cdot}33$ for temporal damping of a standard linear solid. This value of Δ leads, using the above results, to $|\delta\Delta| \simeq 0{\cdot}71$. On the other hand, damping of a plane wave travelling in such a material ($W_0 = 0$) leads for $q = 1$ to $\Delta_{\max} \simeq 1{\cdot}08$ and to $|\delta\Delta| \simeq 0{\cdot}46$. The sum of these two $|\delta\Delta|$ values is 1·17, satisfactorily larger than the actual difference between the two Δ's, $\sim 0{\cdot}25$. Thus, it is clear that the differences found between theoretically calculated temporal and spatial values of Δ are within the expected range. It is evident that considerable care must be exercised in the interpretation of experimentally found Δ values when $\Delta \gtrsim 1$.

Finally, it has been already mentioned that there are situations when the attenuation of a plane sound wave travelling in a material may be intrinsically negative even though the material is non-resonant (Hutson, McFee and White 1961, White 1962). Most of the present analysis should apply to such amplifying systems after a change of sign of α and corresponding changes or reinterpretation of some of the formulae and definitions herein.

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