$$K_1 b_m = 0$$

$$\frac{B_{00}^2}{B_b^2} = 0$$

at $\alpha^2 = 1$. The ordinate at point p is then

$$\frac{B_0^2}{B_b^2} = 2,$$

which determines B_0 and also gives $K_1 q_m$ through the relationship

$$K_1 q_m = \frac{B_0^2}{2B_b^2}.$$

Then K_1 is determined by the known (small) value of q_m shown by the narrow strip in Fig. 6. By use of (76) this value of K_1 yields, in turn, the period L.

In order now to obtain the final Mathieu parameter a, we look for the solid curve in Fig. 7 having $(B_{00}/B_b)^2 = 0$ and passing through point p. In the above example, such a solid curve would be that for $K_1 a_m = 4$. This specifies the value of a. Fig. 6 then shows whether or not this value of a corresponds to a stable region.

Periodic Electrostatic Field

In (1) let

$$B_z = B_{00} = 0 \qquad (77)$$

$$V = V_0 + \Delta V \sin \frac{2\pi}{L} z; \qquad V_0 \gg \Delta V \qquad (78)$$

and

$$r = W V^{-1/4}. \qquad (79)$$

We have then as the final differential equation:

$$\widehat{W}'' + (a_e + 2q_e \cos 2Z)\widehat{W} = -W_1[2q_e \cos 2Z + b_e]. \qquad (80)$$

Where

$$W = W_1 + \widehat{W} \qquad W_1 \gg \widehat{W} \qquad (81)$$

$$W_1 \cong r_1 V^{1/4} \qquad (82)$$

$$a_e = \frac{1}{2}\left(\frac{L}{\lambda_p}\right)^2 + \frac{3}{32}\left(\frac{\Delta V}{V_0}\right)^2 \qquad (83)$$

$$2q_e = \frac{3}{32}\left(\frac{\Delta V}{V_0}\right)^2 \qquad (84)$$

$$b_e = \frac{1}{2}\left(\frac{L}{\lambda_p}\right)^2 - \frac{3}{32}\left(\frac{\Delta V}{V_0}\right)^2. \qquad (85)$$

Eq. (80) is of the same form as (58). Again, the self-consistent solution of (80) is obtained under the conditions

$$b_e = 0 \qquad (86)$$

$$0 < q_e \ll 1. \qquad (87)$$

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The Charging and Discharging of Nonlinear Capacitors*

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***Summary*—The charging and discharging of two types of nonlinear capacitances through a linear resistance are discussed in detail. The response of a capacitor whose capacitance is an increasing exponential function of the potential across it is compared to that of the "space-charge" capacitor whose voltage dependence is of the form $C_s = (C_0 \sinh \alpha V_c/(\alpha V_c))$, where V_c is the potential across the capacitor. The variation of the differential capacitance of the space-charge capacitor with time, during charging and discharging, is considered for various applied potentials and is compared with the somewhat similar behavior to be expected from a linear capacitor exhibiting a distribution of relaxation times.**

INTRODUCTION

THE RAPIDLY growing importance of semiconductor circuit elements with their inherent voltage nonlinearities makes it worthwhile to investigate some of the results of such nonlinearity. Nonlinearity may or may not be of importance in semiconductors depending upon whether diffusion and/or recombination effects dominate the convection current. Although the full accurate equations describing charge-carrier concentration in semiconductors are nonlinear and so have not been solved accurately for all cases of physical interest, there arise many situations where the nonlinearity may be neglected and only the normal linear "metallic" conduction current need be considered.

In the present work, we shall be concerned with the charging of voltage-dependent capacitances through a linear resistance. One of the authors has shown[1] that if an applied direct potential V_0 causes free charge carriers to build up a space charge at a blocking or rectifying electrode, the resulting static capacitance, C_s, defined as q_m/V_0, is of the form $\sinh \alpha V_0/\alpha V_0$, and the differential

* Original manuscript received by the IRE, June 16, 1954, revised manuscript received, November 4, 1954.
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[1] J. R. Macdonald, "Static space-charge effects in the diffuse double layer," *Jour. Chem. Phys.*, vol. 22, pp. 1317–1322; August, 1954.

or small signal capacitance, C_d, given by dq_m/dV_0, is proportional to cosh αV_0. Here q_m is the surface charge on the electrode near which the internal space charge forms. The constant α is $e/4kT$ when both electrodes are blocking, $e/2kT$ when space-charge forms at only one electrode, as for example, when one electrode applied to a semiconducting slab is blocking for positive and negative charge carriers and the other is ohmic. The quantity e is the electronic charge, k Boltzmann's constant, and T the absolute temperature. These results are correct only after V_0 has been applied sufficiently long for space-charge equilibrium to be attained.

Blocking electrodes may occur naturally or be artificially produced at the surface of any material which contains free charge carriers. A rectifying electrode biassed in the reverse direction may also be considered to be blocking for this polarity. Thus, space-charge nonlinearity may be expected to appear under certain conditions in semiconductors,[2,3] photoconductors,[4] and solid or liquid electrolytes.[3,5] Such nonlinearity may sometimes be masked, however, by additional linear capacitance in series with one or more electrodes,[4] or, as in the case of reverse-biassed p-n junctions, additional factors must be taken into account which usually keep the voltage dependence of the junction capacitance from being of the above form or at least restrict such dependence to very small applied voltages.[2,4] It is, of course, obvious that the above very rapid increase of capacitance with applied voltage cannot continue indefinitely as the voltage is increased. Eventually, the internal electric field strength arising from the space-charge distribution near the blocking electrode will become sufficiently high to cause dielectric breakdown of the underlying material or high-field emission at the electrode. Such processes will usually occur at relatively low voltages of the order of a volt or less.[1] In spite of this restriction, nonlinear capacitances of the type discussed in the present work might be well suited for certain switching applications and for use in dielectric amplifiers.

The problem of the time-variation of the charging or discharge current of a material containing free charge, with one or two blocking electrodes, has not been solved exactly because of the nonlinearity of the governing equations. Jaffé and LeMay[6] have treated the problem by linearizing the equations and then attempting to correct the linear solution for the nonlinearity of the equations. This is a very approximate procedure, however, and the final results do not exhibit the strong voltage-dependent nonlinearity to be expected from the nonlinearity inherent in the equations. When a constant potential is applied to a material with blocking or rectifying electrodes, the space-charge capacitance builds up by the motion of charges, leading to the separation of positive and negative charge and the establishment of an excess or deficit of charges of one or the other sign at one or both electrodes. Because such charging is governed by nonlinear equations, a normal "constant" time constant applying during the charging cannot be defined since any time constant will depend on the potential across the material. Although we cannot treat this charging problem exactly, we can treat the related problem of charging of such a material through an external linear resistance much larger than the ordinary internal resistance of the material without blocking, which is $R_i = L[eA(n\mu_n + p\mu_p)]^{-1}$. Here n and p are the concentrations of negative and positive charge carriers and μ_n and μ_p are their respective mobilities. The quantities A and L are the area and separation of the two electrodes, assumed plane and parallel.

When the current which establishes the space-charge distribution must flow through a sufficiently large external resistance, the final equilibrium space-charge distribution corresponding to the actual time-dependent potential difference between the electrodes can be almost established before this potential difference can change appreciably. Under such quasi-static conditions, the time-dependent capacitance will be very nearly given by the static capacitance corresponding to the potential difference actually present at the given time. This potential difference will, of course, be less than that applied to the combination of linear resistance and the charge-containing material until charging is finally complete.

While the above method of treatment of the composite system will be more and more accurate the larger the external resistance, it is not possible to set limits of accuracy in the absence of an exact treatment of the charging of the material itself with no external resistance, R_e. The results of the present analysis indicate, however, that the use of the static capacitance in place of an unknown quasi-static capacitance is probably a fairly good approximation even if R_e is as small as R_i.

It turns out that the above treatment yields results quite similar to those obtained for the charging through a linear resistance of a nonlinear condenser whose capacity is an increasing exponential function of the potential across it. We shall, therefore, analyze this case first, then compare with it the behavior of the nonlinear "space charge" capacitor. It is worth mentioning that the above "exponential" capacitor could be realized, at least over a limited range of applied potential by means of a feedback amplifier of very high gain. If the output is connected to the input through a capacitance C, then the input capacitance of the unit can be approximately $(1+G)C$, where G is the loop gain. This gain is arranged to be linearly proportional to some control voltage. Now

[2] W. Shockley, "The theory of p-n junctions in semiconductors and p-n junction transistors," *Bell Sys. Tech. Jour.*, vol. 28, pp. 435–489; July, 1949.

[3] J. R. Macdonald, "Theory of a-c space-charge polarization effects in photoconductors, semiconductors, and electrolytes," *Phys. Rev.*, vol. 92, pp. 4–17; October, 1953.

[4] J. R. Macdonald, "Capacitance and conductance effects in photoconducting alkali halide crystals," *Jour. Chem. Phys.*, to be published.

[5] J. R. Macdonald, "Theory of the differential capacitance of the double layer in unadsorbed electrolytes," *Jour. Chem. Phys.*, vol. 22; November, 1954.

[6] G. Jaffé and C. Z. LeMay, "On polarization in liquid dielectrics," *Jour. Chem. Phys.*, vol. 21, pp. 920–928; May, 1953.

if a sample of the applied input voltage V_i is used to actuate an electronic computer whose output voltage V_0 is proportional to $(V_i/|V_i|)\exp[\alpha|V_i|]$, and V_0 is then used to control the gain G, we obtain a complicated composite unit whose input capacitance is an exponential function of its input voltage as long as G is considerably greater than unity. The range of variation of the input capacitance can, however, never be larger than the maximum loop gain available.

Mathematical Results

Since we are dealing in this work with nonlinear systems, the principle of superposition[7] will not hold. Therefore, the time dependence of charging currents will differ from that of the corresponding discharge currents,[8] and the two cases must be considered separately.

The system which we shall consider is shown in Fig. 1.

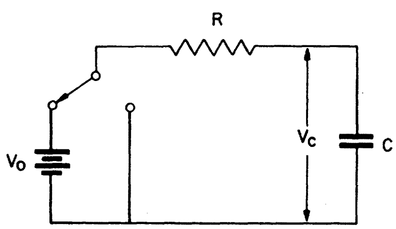

Fig. 1—Circuit for charging and discharging the nonlinear capacitor C through the linear resistor R.

At $t=0$, the resistance is connected to the source of direct voltage V_0, and the completely discharged capacitor begins to charge. The pertinent equation is therefore

$$V_0 - V_c = iR = R\frac{d}{dt}(C_sV_c). \tag{1}$$

For the static capacitance of the exponential capacitor, we take

$$C_s = C_0 e^{\alpha|V_c|}, \tag{2}$$

where α is a positive constant. Thus, (1) becomes

$$V_0 - V_c = RC_0 e^{\alpha|V_c|}\left\{\frac{dV_c}{dt} + \alpha V_c \frac{d|V_c|}{dt}\right\}. \tag{3}$$

Note that a change in sign of all the voltages has no effect on this equation. Therefore, we shall drop the absolute value signs and deal only with positive voltages.

The solution of (3) is presented in Appendix I-A. The result is

$$\tau = e^{-\eta} - e^{-W} + (1+\eta)[Ei(-\eta) - Ei(-W)], \tag{4}$$

where $\tau = t/RC_\infty$, $\eta = \alpha V_0$, $W = \eta - \alpha V_c \equiv \eta - x$ and $Ei(\xi)$ is the exponential integral defined in Appendix I-A. This expression relates the charging time τ, expressed in terms of the final time constant $T_\infty = RC_\infty$, to the instantaneous value of V_c for given V_0 and α. Note that the normalized charging current i/i_0, when expressed in terms of the quantities η and W, is W/η; its time dependence may be obtained from (4). The initial current i_0 is just V_0/R.

Log-log curves computed from (4) for several values of η are presented in Fig. 2. The linear curve is for an ordinary voltage-independent capacitor. We see that when η is large, only a small fraction of T_∞ is required for the capacitor to reach almost its final potential. The reverse is the case, however, if we measure time in terms of initial time constants, $T_0 = RC_0$. Table I gives a comparison of the time required to reach 0.90 V_0, for various values of η, expressed both in terms of t/T_∞ and t/T_0.

TABLE I

Normalized Times Required for V_c/V_0 to Reach 0.90 During Charging for Various Values of η

η	t/T_∞	t/T_0
Linear	2.303	2.303
1	2.670	7.25
4	2.841	155
10	2.042	4.50×10^4
10^2	0.000375	1.01×10^{40}

The differences are due, of course, to the increase in C_s during charging. We shall consider the dependence of charging current in connection with discharge current.

[7] M. F. Gardner and J. L. Barnes, "Transients in Linear Systems," John Wiley and Sons, Inc., New York, N. Y.; 1942.

[8] Note that with only an ac voltage applied, the difference between the charging and discharging currents will show up as hysteresis, a common phenomenon in nonlinear systems.

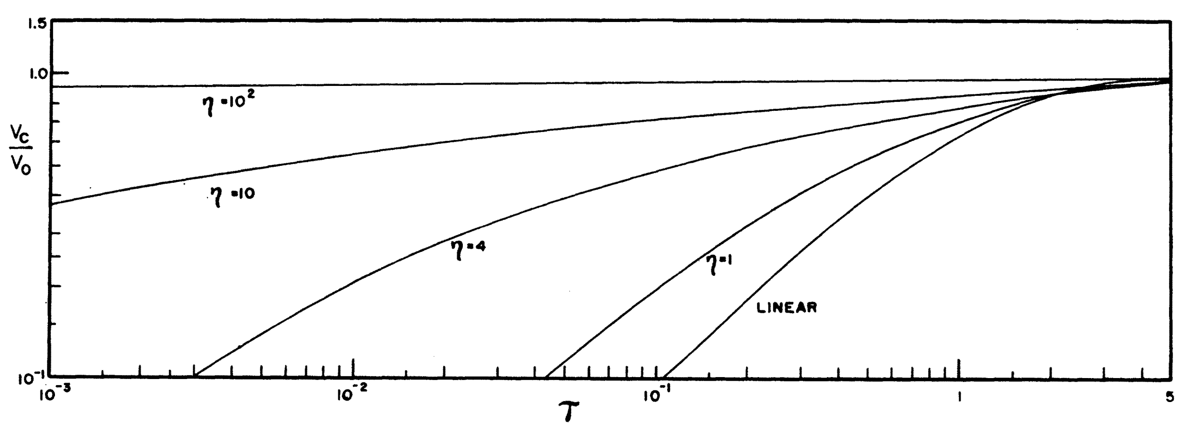

Fig. 2—Log-log plot of the charging of the exponential capacitor for various values of the applied voltage parameter $\eta = \alpha V_0$. The linear curve is for a normal voltage-independent capacitor.

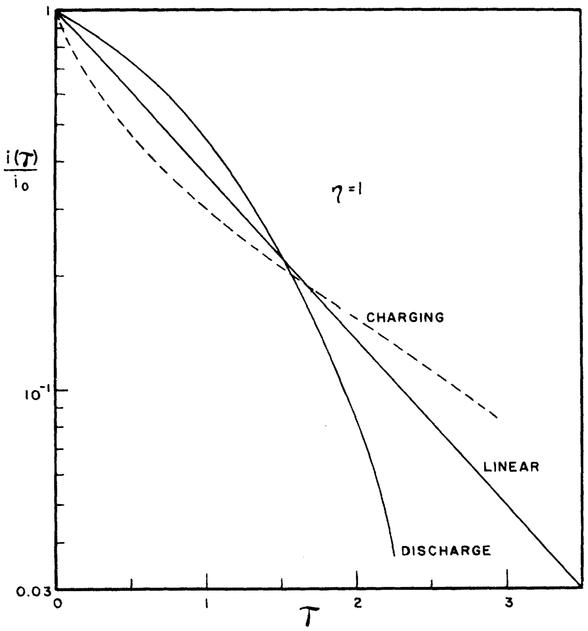

Fig. 3—Semi-log plot of normalized charging and discharge currents versus normalized time for the exponential capacitor with $\eta=1$ and for a linear capacitor.

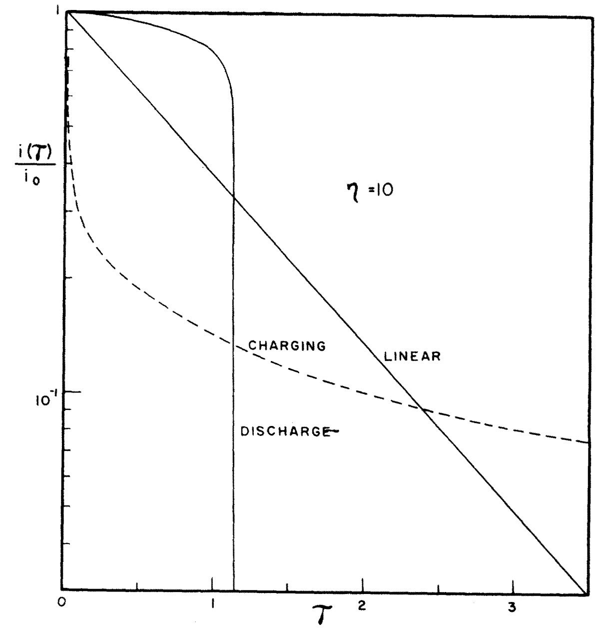

Fig. 5—Semi-log plot of normalized charging and discharge currents versus normalized time for the exponential capacitor with $\eta=10$ and for a linear capacitor.

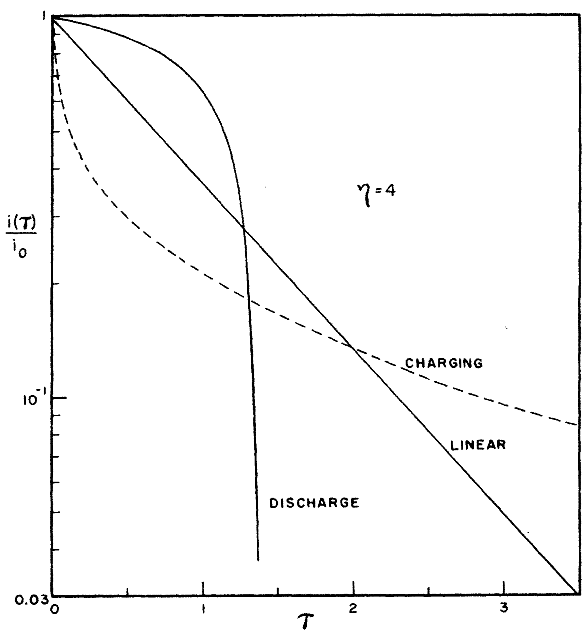

Fig. 4—Semi-log plot of normalized charging and discharge currents versus normalized time for the exponential capacitor with $\eta=4$ and for a linear capacitor.

For discharge, the switch in Fig. 1 is thrown to connect the resistance to ground and the fully charged capacitor begins to discharge. The pertinent equation is

$$V_c = -R\frac{d}{dt}(C_sV_c), \tag{5}$$

which becomes for the exponential capacitor,

$$V_c = -RC_0e^{\alpha|V_c|}\left\{\frac{dV_c}{dt} + \alpha V_c\frac{d|V_c|}{dt}\right\}. \tag{6}$$

The solution of this equation is given in Appendix I-B. The result obtained is

$$\tau = 1 - e^{-W} + J(\eta, W), \tag{7}$$

where $J(\eta, W)$ is an integral defined in Appendix I-B which does not appear to be tabulated in the literature. The results of a graphical evaluation of this integral are tabulated for a few values of η in Table II. (See p. 78.)

We have used (4) and (7) to calculate the decay curves presented in Figs. 3 to 5. The dotted lines are charging curves and represent $(1-V_c/V_0)$ as well as i/i_0. The straight solid lines are the charging and discharging curves for an ordinary linear capacitor, and the curved solid lines are the discharge curves for the exponential capacitor. They represent V_c/V_0 as well as i/i_0.

These results indicate, as expected, that when $\eta \ll 1$ there will be no appreciable difference between the charging and discharging curves of the exponential capacitor and those of an ordinary linear capacitor. In this limit, the exponential capacitor is essentially linear. However, when η becomes much greater than unity, the nonlinearity shows up strongly and the capacitor reaches nearly its final potential in a small fraction of T_∞ on charging; on discharge, however, it discharges at almost constant current for a time of approximately T_∞, then the current falls rapidly to zero. In the limit of very high η, the current will remain constant up to T_∞, then fall abruptly to zero. These results indicate clearly how the essential nonlinearity of the device results in strong differences between charging and discharging behavior.

For the space-charge capacitor, we find on substituting $C_s = (C_0 \sinh \alpha V_c)/(\alpha V_c)$ into (1) and simplifying,

$$V_0 - V_c = RC_0[\cosh(\alpha V_c)](dV_c/dt). \tag{8}$$

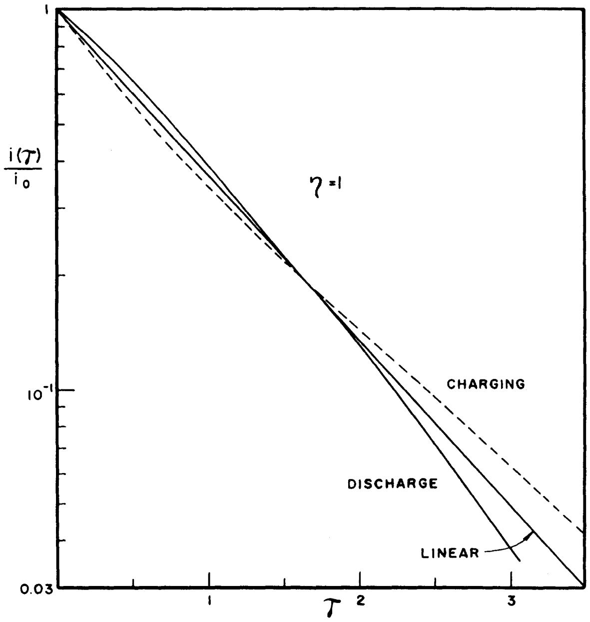

Fig. 6—Semi-log plot of normalized charging and discharge currents versus normalized time for the space-charge capacitor with $\eta = 1$ and for a linear capacitor.

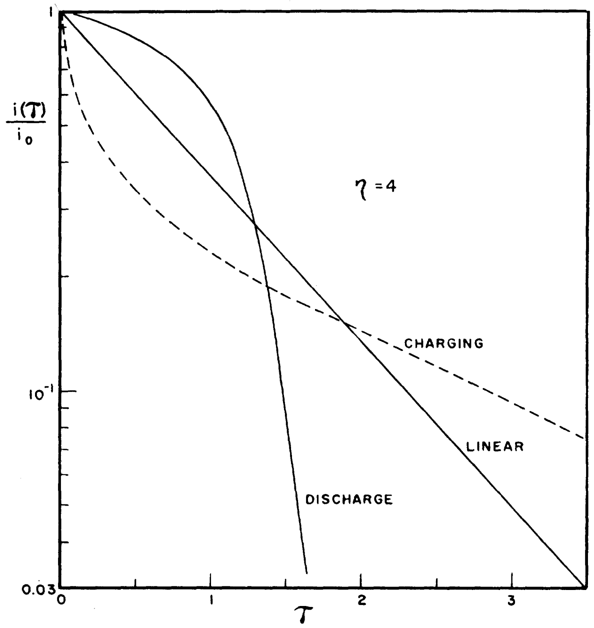

Fig. 7—Semi-log plot of normalized charging and discharge currents versus normalized time for the space-charge capacitor with $\eta = 4$ and for a linear capacitor.

The solution of this eq. is found in Appendix II-A to be

$$\tau = [\eta e^{\eta}/(2 \sinh \eta)][Ei(-\eta) - Ei(-W) + e^{-\eta}J(\eta, x)], \quad (9)$$

where $J(\eta, x)$ is again the integral defined in (10) of Appendix I-B.

In a similar fashion, using (5), we find that the discharge equation for the space-charge capacitor is (see Appendix II-B)

$$\tau = [\eta e^{\eta}/(2 \sinh \eta)] \\ [e^{-\eta}\{Ei(-\eta) - Ei(-x)\} + J(\eta, W)]. \quad (10)$$

Eqs. (9) and (10) exhibit a symmetry not apparent in the corresponding exponential capacitor equations.

Figs. 6 and 7 present the charging and discharge curves of the space-charge capacitor for $\eta = 1$ and 4. On comparing these curves with those of Figs. 3 and 4 for the exponential capacitor, one sees that the nonlinearity is not quite so apparent for corresponding values of η for the space-charge as for the exponential capacitor. However, for η-values of 100 or greater, there is essentially no difference between corresponding curves. All of these conclusions are, of course, consistent with the forms of the dependence of capacitance on applied potential for the two types.

Discussion

The present theory predicts that the discharge current of a charged material containing free charge carriers blocked at at least one electrode may, for large η, remain almost constant for some time, then fall rather abruptly to zero. Experimentally, however, it is sometimes found that the discharge current of a polarized material begins to increase above its initial value, reaches a maximum, and then decays. Such behavior probably arises from the motion of a number of carriers of different mobility as well as probable failure of the quasi-static approximation of the present work. The experimental points and solid line of Fig. 8 represent, however, an experimental current discharge curve similar to those predicted by the present theory. The current is given here in arbitrary units. This curve was obtained by one of the authors (J. R. Macdonald) several years ago on a KBr single crystal containing F-centers. The principal charge carriers present were probably positive-ion vacancies and electrons, although negative-ion vacancies and positive ions may also have contributed

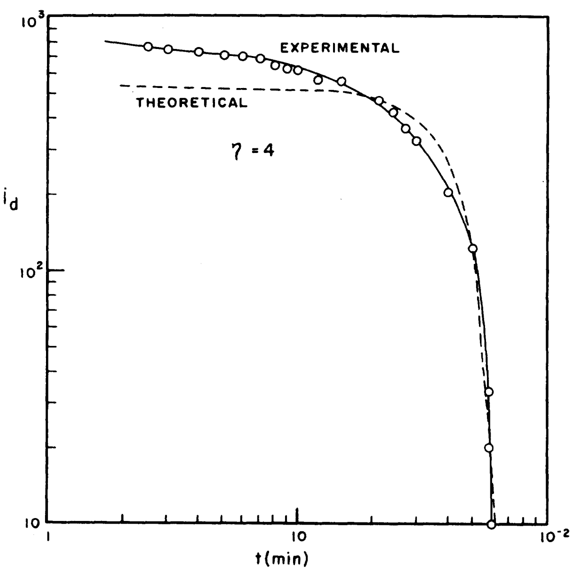

Fig. 8—Log-log plots of experimental and theoretical (space-charge capacitor) discharge curves.

slightly to the discharge current. In order to eliminate rapidly decaying transients such as that arising from the geometrical capacitance, the crystal was first shorted momentarily after charging for ten minutes, then measurement of the discharge current was begun.

The dotted curve of Fig. 8 is the $\eta=4$ discharge curve for the space-charge capacitor, replotted on this log-log graph in a position in best agreement with the experimental curve. Such positioning is valid for a log-log presentation. While agreement between theory and experiment is by no means perfect (it could be somewhat improved by a slightly smaller value of η), the degree of correspondence is sufficient to suggest that the theory can predict some of the experimental features ressonably well. As discussed above, most of the disagreement can probably be ascribed to partial failure of the quasistatic approximation.

It is perhaps worth mentioning that discharge curves on these crystals often exhibited a current reversal following a rapid decay like that shown in Fig. 6. The approximate theory of Jaffé and Lemay predicts such a reversal on discharge, although their predicted curve shapes do not agree well with those observed on the KBr crystals. Such disagreement is not surprising since these authors carried out the second approximation to their initially linearized theory by tacitly assuming the validity of the principle of superposition. It is this second approximation which they find leads to the current reversal.

There are a number of possibilities which might explain such a reversal. If a nonlinear capacitor were first charged with one polarity, only partly discharged, and then recharged with the opposite polarity, the discharge current then measured might reverse because of the complicated charging history. Reversal could certainly be observed for a linear capacitor exhibiting a distribution of relaxation times[9] charged in this fashion. Although the details would be different for a nonlinear capacitor, there is no reason not to expect a reversal in this case also, even though the principle of superposition fails. It seems unlikely that an exact theory of the discharge of a free-charge-containing material, containing positive and negative charges of equal mobility, would yield a current reversal for normal unipolar charging. The reversal of the Jaffé-Lemay theory may possibly arise from the approximate character of the theory. If this conclusion is valid, then an entirely different mechanism must be called into play to explain the observed reversal. We suggest that this mechanism might, in some cases, be the polarization of the underlying medium by the high fields produced by the space-charge distribution.[1] Since this polarization will be opposite in direction to that represented by the space-charge distribution itself, if the space-charge decays much faster than the polarization of the medium, there will be a reversal of the total current when the faster decaying process decays to a smaller current value than the slower reverse-current process. Note that the decay of the polarization of the medium alone may be expected to be a single exponential decay or, in any event, a sum of exponential decaying terms involving voltage-independent time constants rather than the voltage-dependent decay characteristic of a nonlinear capacitor.

It is sometimes impractical to measure either the charge or discharge current of a nonlinear capacitor because of the smallness of these currents, or because of the presence of shunting resistance. In such cases, however, the differential small-signal capacitance may often be measured during the charging or discharging. Then, from such a measurement, it may be possible to draw conclusions regarding the presence or absence of nonlinearity. The differential capacitance of the space-charge capacitor is

$$C_d \equiv C_s + V_c \frac{dC_s}{dV_c} = C_0 \cosh \alpha V_c.^{5,10}$$

It is measured by means of an ac signal much smaller in amplitude than the dc charging voltage across the capacitor. If an ac voltage V_{ac} so small that αV_{ac} is always much less than unity is employed, the initial capacitance C_0 is obtained when the dc charging potential V_c is zero. We shall assume this to be the case.

We are now interested in the time dependence of the readily measurable quantity C_d during charging and discharging for various values of η. Since $C_d/C_0=\cosh x$, we can easily calculate such results using (9) and (10). The results for $\eta=4$ and 10 are presented in Figs. 9 and 10. On log-log plots of this nature, we see that there is a considerable time interval during which the slopes of the charging curves are less than unity and are almost constant. As η becomes very large, the slope in this interval approaches unity, while it approaches zero for $\eta \ll 1$. These curves for capacitance time-dependence again demonstrate the strong nonlinearity of the space-charge capacitor for large η.

Unfortunately, nonlinearity of the type which we have been considering is not the only factor which can cause the apparent capacitance of a capacitor to vary with time during charging and discharging. If the material of which the capacitor is composed contains no free charges but does exhibit a distribution of relaxation times, then the apparent total capacitance will increase during charging as elements with longer and longer relaxation times are charged. Since it is commonly assumed that the microscopic processes which lead to a relaxation-time distribution are independent of one another,[11] the principle of superposition still holds, the material is linear, and the static and differential capacitances are equal. In addition, unlike the nonlinear capacitor, the charging and discharging currents of a

[9] J. R. Macdonald, "Dielectric dispersion in materials having a distribution of relaxation times," *Jour. Chem. Phys.*, vol. 20, pp. 1107–1111; July, 1952.

[10] D. C. Grahame, "The electrical double layer and the theory of electrocapillarity," *Chem. Rev.*, vol. 41, pp. 441–501; December, 1947.

[11] H. Frohlich, "Theory of Dielectrics," Clarendon Press, Oxford, Eng., p. 91; 1949.

capacitor with a distribution of relaxation times should be the same if there is no shunting leakage of charge.

An example of a circuit involving a distribution of relaxation times is afforded by the parallel connection of an arbitrary number of series branches, each branch

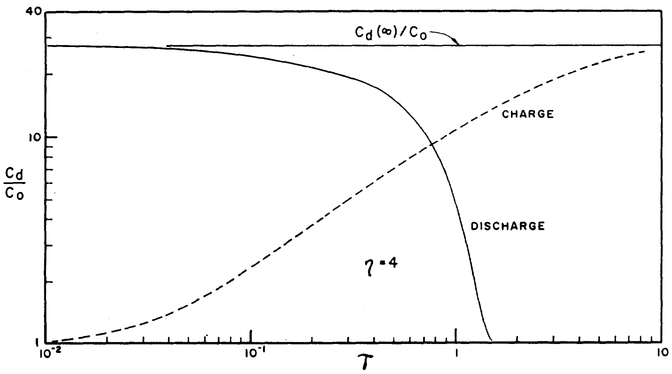

Fig. 9—Log-log plots of the time dependence of the differential capacitance of the space-charge capacitor for $\eta=4$ during charging and discharging.

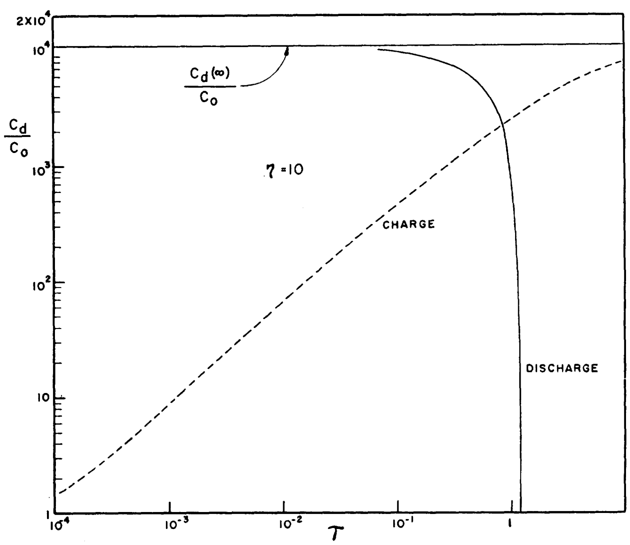

Fig. 10—Log-log plots of the time dependence of the differential capacitance of the space-charge capacitor for $\eta=10$ during charging and discharging.

consisting of a linear capacitance C_i in series with a linear resistance R_i. The time constant of each branch is then $\tau_i = R_iC_i$. Since the circuit is entirely linear, a calculation of the apparent static capacitance also gives the differential capacitance. Using the definition that the capacitance at time t is the charge stored in the system at t divided by the applied voltage, a simple calculation yields

$$C_d(t) = \sum_i C_i(1 - e^{-t/\tau_i})$$

$$R_d(t) = \left\{ \sum_i R_i^{-1} e^{-t/\tau_i} \right\}^{-1},$$

where $R_d(t)$ is the apparent resistance at time t, defined as the applied voltage divided by the current at t. As expected, these equations predict that the over-all measured capacitance increases from zero to the final value $\sum_i C_i$, and the apparent resistance increases from $\left\{ \sum_i R_i^{-1} \right\}^{-1}$ to the final value infinity. Any shunt resistance will make the final resistance value finite.

In Fig. 11, we present the time variation of differential capacitance measured on a large-area silicon p-n junction. Time was measured from the instant that a

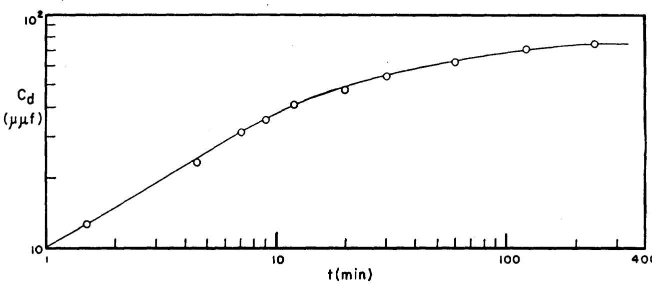

Fig. 11—Log-log plot of the time dependence of the differential capacitance of a large-area silicon p-n junction with 3 volts reverse bias applied at $t=0$.

reverse bias of 3 volts was applied to the junction. The differential capacitance of the junction was measured with an ac signal of 0.1 volt rms or less. This curve is quite similar in character to the charging curve of Fig. 9 for $\eta=4$, applying to the nonlinear space-charge capacitor. Since a semiconductor does contain free charges, most of which are blocked at a reverse-biassed junction, it was initially thought that the curve of Fig. 9 did indeed represent nonlinear behavior arising from the motion of these charges. Since the value of η which gives a reasonable fit of the curve on this assumption is 3 or 4, we find that if α is taken as $(e/4kT)\sim 10$, the actual final potential across the junction could not have exceeded about 0.4 volt, instead of the 3 volts applied. This conclusion is not as untenable as it seems, because it might be explained as arising from voltage division between a series charging resistance in the material and a shunt resistance across the junction.

The above hypothesis was definitely shown to be wrong, however, by varying the reverse bias voltage over a wide range. Such variation should produce even greater changes in C_∞ and in the slope of the C_d-versus-time curve if the material is strongly nonlinear; instead, a large increase in the reverse bias had only a small effect on the differential capacitance. Therefore, the curve of Fig. 9 cannot arise from an ordinary nonlinear space-charge process. By heating the junction for several hours at 150 degrees C., then encapsulating the unit in a container filled with silicone oil, the effect could be greatly reduced. This result indicates that it is probably almost entirely a surface phenomenon. It is quite possible that it arises from a distribution of relaxation times of surface contaminants near the junction, perhaps connected with a distribution of surface states. This effect is probably analogous to that which leads to channeling in germanium n-p-n junction transistors.[12] It thus appears that the two distinct processes, space-charge formation and charging of a material with a distribution of relaxation times, can lead to quite similar results in cer-

[12] W. L. Brown, "N-type surface conductivity on p-type germanium," *Phys. Rev.*, vol. 91, pp. 518–527; August, 1953.

tain cases and that therefore other criteria than the time dependence of C_d may be required to distinguish between the two.

Appendix I—The Exponential Capacitor

A. Charging

If we drop the absolute value signs in (3) of the text, consider only positive voltages, and introduce the normalized quantities

$$\begin{aligned} \eta &= \alpha V_0 \\ x &= \alpha V_c \\ \tau &= t/RC_\infty, \end{aligned} \tag{11}$$

(3) becomes

$$\frac{dx}{d\tau} = \left(\frac{\eta - x}{1 + x}\right) e^{(\eta - x)}, \tag{12}$$

where we have used $C_\infty = C(\infty) = C_0 e^{\alpha V_0} = C_0 e^\eta$. Separating and integrating this equation from 0 to τ, we obtain

$$\tau = \int_0^x \left(\frac{1 + y}{\eta - y}\right) e^{-(\eta - y)} dy. \tag{13}$$

If we introduce the quantities $w = \eta - y$ and $W = \eta - x$ in (13), we obtain

$$\tau = (1 + \eta) \int_W^\eta \frac{e^{-w}}{w} dw - \int_W^\eta e^{-w} dw. \tag{14}$$

Now recalling that the exponential integral $Ei(-\xi)$, tabulated *e.g.* by Jahnke and Emde,[13] is defined as

$$Ei(-\xi) = -\int_\xi^\infty \frac{e^{-w}}{w} dw, \tag{15}$$

(14) becomes

$$\tau = e^{-\eta} - e^{-W} + (1 + \eta)[Ei(-\eta) - Ei(-W)]. \tag{16}$$

B. Discharge

On expressing (6) of the text in terms of normalized variables, we obtain

$$\frac{dx}{d\tau} = -\left(\frac{x}{1 + x}\right) e^{(\eta - x)}. \tag{17}$$

Separating and integrating, we find

$$\begin{aligned} \tau &= -\int_\eta^x \left(\frac{1 + y}{y}\right) e^{-(\eta - x)} dy \\ &= 1 - e^{-W} + \int_x^\eta \frac{e^{-(\eta - y)}}{y} dy \\ &= 1 - e^{-W} + J(\eta, W), \end{aligned} \tag{18}$$

where the integral $J(\eta, \xi)$ is defined as

$$J(\eta, \xi) = \int_{\eta - \xi}^\eta \frac{e^{-(\eta - y)}}{y} dy = \int_0^\xi \frac{e^{-w}}{\eta - w} dw. \tag{19}$$

This integral is a function of two variables and cannot, unfortunately, be expressed in terms of exponential integrals. So far as the authors are aware, it is untabulated in the literature.

It is, however, a simple matter to evaluate $J(\eta, \xi)$ graphically for the values of η in which we are most interested. The results are presented in Table II. The values in this table are probably not accurate to more than three decimal places.

TABLE II
The Integral $J(\eta, \xi)$

ξ/η \ η	1	4	10
0	0	0	0
0.1	0.10018	0.08625	0.06788
0.2	0.20150	0.15079	0.09463
0.3	0.30537	0.20001	0.10578
0.4	0.41376	0.23803	0.11063
0.5	0.52980	0.26793	0.11268
0.6	0.65892	0.29258	0.11348
0.7	0.80692	0.31391	
0.8	1.0085	0.33399	
0.91	1.3169	0.35682	
0.95	1.5797	0.37387	

Appendix II—The Space-Charge Capacitor

A. Charging

In terms of normalized variables, the equation which must be solved is

$$\frac{dx}{d\tau} = \left(\frac{\eta - x}{\cosh x}\right)\left(\frac{\sinh \eta}{\eta}\right), \tag{20}$$

where we have used $C_s(\infty) = C_0 \sinh \eta/\eta$ and

$$\tau = t/RC_s(\infty) = t/T_\infty.$$

The above equation leads to the integral

$$\tau = \frac{\eta}{2 \sinh \eta} \int_0^x \left(\frac{e^x}{\eta - x} + \frac{e^{-x}}{\eta - x}\right) dx. \tag{21}$$

The first part of this integral is easily expressed in terms of the exponential integral, while the latter part is, from (19), just $J(\eta, x)$. We therefore obtain

$$\tau = [\eta e^\eta / 2 \sinh \eta][Ei(-\eta) - Ei(-W) + e^{-\eta} J(\eta, x)]. \tag{22}$$

B. Discharge

The discharge equation reduces to

$$\frac{dx}{d\tau} = -\left(\frac{x}{\cosh x}\right)\left(\frac{\sinh \eta}{\eta}\right). \tag{23}$$

The resulting integral is

$$\tau = \frac{\eta}{2 \sinh \eta} \int_x^\eta \left[\frac{e^y}{y} + \frac{e^{-y}}{y}\right] dy. \tag{24}$$

The latter part of this integral may be readily expressed in terms of exponential integrals. On making the substitutions $W = \eta - x$ and $w = \eta - y$ in the first part of the integral, it reduces to $e^\eta J(\eta, W)$. The discharge relation therefore becomes

$$\tau = [\eta e^\eta / 2 \sinh \eta][e^{-\eta}\{Ei(-\eta) - Ei(-x)\} + J(\eta, W)]. \tag{25}$$

[13] E. Jahnke and F. Emde, "Tables of Functions," Dover Publications, New York, N. Y., pp. 1–8; 1943.