Linear-System Integral Transform Relations

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I. INTRODUCTION

THERE are a number of quantities which may be used to characterize a linear system. In this work we shall be concerned both with the problem of how much information about the system each such pertinent quantity can yield and with the interrelations between these quantities. A central part of the discussion will deal with the integral transform relationships between the real and imaginary parts of a complex function of frequency which contains complete information concerning the behavior of the system after excitation with an arbitrary stimulus. This complex function may be an input or transfer impedance or admittance, a transfer ratio, a complex susceptance, etc.

A great many fields of science involve linear systems or nonlinear systems whose deviation from linearity may be neglected. The fundamental importance of the linear system has ensured that many of the relations with which we shall be concerned have long been known. It is our aim to give a consistent and reasonably complete derivation and discussion of the older relations and to show how useful new relations may be generated. The recent appearance of extensive tables of integral transforms[1,2*] makes the transform relations with which we shall be concerned of considerable practical usefulness.

There are several reasons why integral transforms are of interest. First, mathematical operations with a transformed function may often be simpler than with the original function. For example, the operation of differentiation of an original function is replaced by a multiplication process when the function is Laplace transformed.[3] Second, various transforms may be employed to change a known function of a variable such as time to the corresponding function of another variable such as frequency. The resulting function may often be more useful than the original function. In addition, an easily measured function such as the response of a system to a forcing step-function may be transformed to yield a desired function such as the distribution of relaxation times of the system which would itself be more difficult to measure directly.

In general, integral transforms of the type we shall consider convert a given function, say $f(z)$ of the variable z, to another function $g(w)$ of the same or different variable. We may write, therefore,

$$g(w)=\int_{C_1} K_1(w,z)f(z)dz,$$

where K is a function of both w and z called the kernel, w and/or z may be complex, and C_1 denotes a path in the complex plane. If $f(z)$ is known and K_1 and C_1 specified, $g(w)$ may then be obtained if the integration can be carried out. On the other hand, the above equation is also a linear integral equation for $f(z)$ when $g(w)$, K_1, and C_1 are specified. If z and w are real variables, it is a Fredholm equation of the first kind. The solution may be formally written as

$$f(z)=\int_{C_2} K_2(z,w)g(w)dw,$$

another integral transform and integral equation.

For each integral transform, there will be another relation which converts the transformed function back to the original function. This relation will also usually be an integral transform itself although it may sometimes be written in terms of algebraic operations only. For all practical purposes, there is unique one-to-one correspondence between a function and its transform and between the transformed function and the original function. When the kernels K_1 and K_2 are equal and the paths of integration the same for the forward and inverse transforms, the relations between $f(z)$ and $g(w)$ are said to be reciprocal, and they are termed conjugate functions. When these quantities are equal save for a minus sign as in the Hilbert transform,[3] the relations are skew-reciprocal.

* References and notes are given at the end of this article.

These general statements may be illustrated specifically by means of the one-sided Laplace transform. If one changes the variables to conform with later usage, the Laplace transform of a function of time $f(t)$ may be written

$$g(p)=\int_0^\infty e^{-pt}f(t)dt,$$

where p is a complex (frequency) variable. The inverse transform is given by

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{tp}g(p)dp \quad (t\geqslant 0),$$

where c is a real constant.[3] It is also worth noting that an algebraic Laplace transform inversion formula is known for obtaining $f(t)$ from $g(p)$ which requires knowledge of $g(p)$ for real values of p.[4]

Integral transform relations of the types discussed herein have been employed in quantum mechanics,[5] in quantum field theory and the quantum theory of scattering,[6–8] viscoelasticity,[9–13] circuit theory,[14] dielectric theory,[15,16] magnetic resonance,[17–19] etc. This is by no means an exhaustive list. The integral relations between the real and imaginary parts of a complex function of frequency such as an impedance are generally known among physicists as the Kronig-Kramers relations.[20,21] Because of their importance, they have appeared and been used in a number of different fields. A rather complete bibliography pertaining to these relations has been given by Murakami and Corrington.[22]

Although everyone feels that he intuitively knows the meaning of the term linearity, it may be well for us to discuss the matter briefly. One of the most fundamental definitions of linearity is that involving the differential (or integral) equation or equations which the system satisfies. If this equation is of the first degree with constant coefficients, it is a linear differential equation and the system described is linear by definition. The response of a linear system to two or more superposed inputs is the sum of the responses to the individual inputs. This property is an immediate corrollary of the fact that the system satisfies a linear differential equation; it is equivalent to the statement that the principle of superposition[23] applies to the system. Note that a linear system is not retroactive and there is no hysteresis. In the present work, we shall restrict consideration to time-invariant linear systems. The structure of such systems is independent of time; therefore, the coefficients of the differential equation(s) of the system are also time-invariant.

II. BASIC RELATIONS

A linear system may be characterized by a detailed description of its parts and their interconnection or by its response to a disturbing force (mechanical force, voltage, current, etc.). The excitation functions most useful for such characterization are the unit impulse,[24,25] the unit step,[26,27] and steady-state sinusoidal excitation.

We may define the system function $S(i\omega)$ as the complex ratio of output to input amplitudes for steady-state sinusoidal excitation of radial frequency ω. More generally, it may be considered as a function $S(p)$ of a complex frequency variable p. $S(p)$ may then take on values over the entire complex p plane instead of only on the $i\omega$ axis of this plane. As we shall see, such a generalization is extremely useful. The system function may represent a driving point function, such as an input or transfer impedance or admittance, or a transfer ratio, depending upon the nature of the exciting force applied to the system and upon the point in the system at which the response is measured. Similarly, let us define $A(t)$ as the system response to a unit step function $u_0(t)$, and $B(t)$ as the response to a unit impulse $\delta(t)$. As is evident, both $u_0(t)$ and $\delta(t)$ are applied at $t=0$. For $t<0$, the system is usually taken to be at rest.

The requirement of system linearity may be expressed mathematically by the principle of superposition. If the response of the system to an excitation $f(t)$ is given by $r(t)$, we may write[23]

$$r(t)=\int_{0-}^{t} f(\tau)B(t-\tau)d\tau \quad (t\geqslant 0),$$
$$=\int_{0-}^{\infty} f(t-\tau)B(\tau)d\tau, \tag{1}$$

and

$$r(t)=f(0)A(t)+\int_0^t \frac{df(\tau)}{d\tau}A(t-\tau)d\tau,$$
$$=\frac{d}{dt}\int_{0-}^{t} f(\tau)A(t-\tau)d\tau \quad (t\geqslant 0). \tag{2}$$

The lower limits of integration in (1) and (2) are written as $0-$ instead of simply 0 to account for the possibility that $f(\tau)$, $B(\tau)$, or $A(\tau)$ involve impulse functions. The quantity $0-$ may be represented by $-\epsilon$, where $\lim \epsilon\to 0$. These real convolution integrals,[4,28] are basic in determining the response of a system to an arbitrary excitation applied at $t=0$. In the first expression, $r(t)$ has been represented by means of a superposition of system responses to delayed unit impulses $\delta(t-\tau)$, whereas in (2) the superposition is carried out for responses to the unit step function $u_0(t-\tau)$. The integrals of Eqs. (1) and (2) may sometimes be difficult to evaluate exactly and approximation methods may have to be employed.[29] When graphical integration is used, for example, determination of $r(t)$ for each selected value of time requires a separate approximate integration. Since the system response to arbitrary excitation may be expressed, as above, in terms of its impulse-function or step-function response, these quantities are of primary importance and fully characterize the system transient response.

The Laplace transforms of the above equations are particularly simple. They are[28]

$$R(p)=\mathfrak{B}(p)F(p), \quad (3)$$

and

$$R(p)=p\mathfrak{A}(p)F(p). \quad (4)$$

In the above equations, $p=\sigma+i\omega$, where σ and ω are real, ω is the radial frequency, and σ is a constant. The quantities $R(p)$, $F(p)$, $\mathfrak{B}(p)$, and $\mathfrak{A}(p)$ are the Laplace transforms of $r(t)$, $f(t)$, $B(t)$, and $A(t)$, respectively.

It is clear that the temporal response of the system can be immediately obtained whenever it is possible to carry out an inverse Laplace transformation of the products in (3) or (4). The form of the equations shows these products to be equal. This method of obtaining the system transient response is also often of great usefulness.

Next, if we set $\mathfrak{B}(p)$ and $p\mathfrak{A}(p)$ equal and take the inverse Laplace transform of the resulting equation, we obtain

$$B(t)=A(0)\delta(t)+\frac{dA(t)}{dt}, \quad (5)$$

a relation between the system responses to unit impulse and to unit step. This equation may be integrated to yield the inverse relationship

$$A(t)=\int_{0-}^{t}B(\tau)d\tau, \quad (6)$$

where the lower limit of integration is again extended to $0-$ to ensure that the range of integration encompasses the region at $t=0$ yielding a contribution from $\delta(t)$. Since $B(t)$ and $A(t)$ may sometimes contain impulse functions, they are not necessarily zero at $t=0$. On the other hand, when system excitation is applied at $t=0$, they are identically zero for $t<0$. For simplicity, we shall generally omit further mention of this fact in the rest of this work. Now Eqs. (3) and (4) represent the Laplace-transformed relations between the response of the system and its excitation. Therefore, $\mathfrak{B}(p)$ and $p\mathfrak{A}(p)$ may each be identified with the system function $S(p)$.[30] Let us introduce the new terminology $\mathfrak{A}(p)\equiv Q(p)$, where $Q(p)$ will be called the network function[31] to distinguish it from the system function. Elsewhere,[32] $Q(p)$ has also been termed an immittance[33] kernel; such nomenclature is, however, not sufficiently general since $Q(p)$ may be related to a system transfer ratio instead of an input or transfer impedance or admittance.

We have now established the relations

$$S(p)\equiv\mathfrak{B}(p)\equiv pQ(p). \quad (7)$$

It is thus evident that the system function is the Laplace transform of the response to unit impulse. Although it is not necessary to imply the usual restricted interpretation of the Laplace transform that the time function is identically zero for $t<0$,[25] it will usually be convenient in the present work to adhere to this restriction. A linear system is not retroactive; if an excitation is applied at $t=0$, there can be no prior response, and thus all pertinent time functions must be zero for $t<0$. This restriction is often called the causality condition. It may be noted that the step-function response, $A(t)$, is often called the indicial admittance[34] or the decay function[35] of the system. On the other hand, the unit impulse response, $B(t)$, is the system Green's function.[36,37] It is also often termed the system weighting function.[38,39] Its weighting character is evident from Eq. (1) which shows that the system output at a given time depends on the input at that time and on that at all earlier times with the weighting of earlier inputs being determined by $B(t)$.

It is often useful to consider the real and imaginary parts of the system and network functions separately in the limit of $\sigma\rightarrow 0$, so that $p\rightarrow i\omega$. To obtain the desired relations, we may begin with the equations

$$S(p)=\int_{0-}^{\infty}B(t)e^{-pt}dt\equiv\mathfrak{L}[B(t)], \quad (8)$$

and

$$Q(p)=\int_{0-}^{\infty}A(t)e^{-pt}dt\equiv\mathfrak{L}[A(t)], \quad (9)$$

where the symbol $\mathfrak{L}$ represents the Laplace transform integral operator. These equations may also be written in the useful inverted forms

$$B(t)=\mathfrak{L}^{-1}[S(p)], \quad (10)$$

$$A(t)=\mathfrak{L}^{-1}[Q(p)], \quad (11)$$

where $\mathfrak{L}^{-1}$ denotes the inverse Laplace transform operator.[3]

These equations show the intimate relationship between the system function expressed in terms of the complex frequency variable p and the temporal response of the system. The alternative possibility of using Fourier instead of Laplace transforms in the above equations will be discussed later.

We may now define the quantities

$$\lim_{\sigma\rightarrow 0}S(p)\equiv S(i\omega)=P(\omega)+iT(\omega), \quad (12)$$

and

$$\lim_{\sigma\rightarrow 0}\left[\frac{S(p)}{p}\right]\equiv Q(i\omega)=J(\omega)-iH(\omega). \quad (13)$$

Note that $S(i\omega)$ is a complex function of the real radial frequency ω. It is the quantity which would be measured with sinusoidal excitation and may represent an input immittance, a transfer ratio, or a transfer immittance. The network function, $Q(i\omega)$, is more of a derived quantity, although it is of great importance. Characteristically, it may represent a complex magnetic or electric susceptance or may be merely related to a transfer ratio specified by the corresponding $S(i\omega)$. If $J(\omega)$ and $H(\omega)$ are proportional to the real and imaginary parts of the complex dielectric constant of a lossy capacitance, for example, it is clear that $P(\omega)$ and $T(\omega)$ are proportional to the real and imaginary parts of the admittance of the system. The relations between the real quantities in (12) and (13) are of particular interest although their derivation is not an easy task. There are four equations connecting the four real quantities of interest. To obtain them, we may write the equations

$$\lim_{\sigma\to 0} S(\sigma+i\omega)=\lim_{\sigma\to 0}[(\sigma+i\omega)Q(\sigma+i\omega)], \qquad (7')$$

and

$$\lim_{\sigma\to 0} Q(\sigma+i\omega)=\lim_{\sigma\to 0}[(\sigma+i\omega)^{-1}S(\sigma+i\omega)]. \qquad (14)$$

Now, using the definitions (12) and (13), rationalizing, and separating into real and imaginary parts, we obtain

$$P(\omega)=\lim_{\sigma\to 0}\{\sigma \operatorname{Re}[Q(\sigma+i\omega)]-\omega \operatorname{Im}[Q(\sigma+i\omega)]\}$$
$$=\omega H(\omega)+\lim_{\sigma\to 0}\{\sigma \operatorname{Re}[Q(\sigma+i\omega)]\}, \qquad (15)$$

$$T(\omega)=\lim_{\sigma\to 0}\{\omega \operatorname{Re}[Q(\sigma+i\omega)]+\sigma \operatorname{Im}[Q(\sigma+i\omega)]\}$$
$$=\omega J(\omega)+\lim_{\sigma\to 0}\{\sigma \operatorname{Im}[Q(\sigma+i\omega)]\}, \qquad (16)$$

$$J(\omega)=\lim_{\sigma\to 0}\left[\frac{\sigma \operatorname{Re}[S(\sigma+i\omega)]+\omega \operatorname{Im}[S(\sigma+i\omega)]}{\sigma^2+\omega^2}\right], \qquad (17)$$

$$H(\omega)=\lim_{\sigma\to 0}\left[\frac{\omega \operatorname{Re}[S(\sigma+i\omega)]-\sigma \operatorname{Im}[S(\sigma+i\omega)]}{\sigma^2+\omega^2}\right]. \qquad (18)$$

Equations (15) through (18) represent the desired connections in a fairly general form. Unfortunately, they still cannot be used in this form without explicit knowledge of the functions $S(\sigma+i\omega)$ and $Q(\sigma+i\omega)$. We have, thus far, failed in finding a rigorous method of simplifying them further. In addition, the rationalization leading to (17) and (18) may not always be allowable, depending on the form of $S(\sigma+i\omega)$.[40] The difficulty arises from the fact that terms of the form $\lim_{\sigma\to 0}[f(\sigma)S(\sigma+i\omega)]$ are not necessarily equal to $\lim_{\sigma\to 0}[f(\sigma)S(i\omega)]$.

To further simplify (15) through (18), we must recall the following representations of the delta or first-order impulse function and its first derivative, the doublet impulse.[40,41]

$$\pi\delta(\omega)=\lim_{\sigma\to 0}[\sigma/(\sigma^2+\omega^2)], \qquad (19)$$

$$\pi\delta'(\omega)=-\lim_{\sigma\to 0}[2\omega\sigma/(\sigma^2+\omega^2)^2]. \qquad (20)$$

First, we see that if terms such as $\sigma \operatorname{Re}[Q(\sigma+i\omega)]$ or $\sigma \operatorname{Im}[Q(\sigma+i\omega)]$ should be proportional to either of the above forms or to higher derivatives of the delta function, these terms would not go to zero in the limit $\sigma\to 0$. We have been unable, however, to find any reasonable forms for $Q(p)$ for which these terms do not vanish, and we suggest without proof that they must always go to zero if $Q(p)$ is an analytic function of the complex variable p. An example is $Q(p)=1/p$. Then, $\lim_{\sigma\to 0}\{\sigma \operatorname{Im}[Q(\sigma+i\omega)]$ is $\pi\omega\delta(\omega)$, which is identically zero. Note that this $Q(p)$ is nonanalytic at $p=0$. Thus, for all practical purposes, we are justified in writing

$$P(\omega)=\omega H(\omega), \qquad (15')$$

$$T(\omega)=\omega J(\omega). \qquad (16')$$

It may be noted that these last results may be alternatively derived from some of our later integral transforms; for example, if we integrate (34) by parts and substitute (23), we obtain (16′) directly.

The simplification of (17) and (18) is more complicated. Here, for example, either of the terms $\sigma \operatorname{Re}[S(\sigma+i\omega)]/(\sigma^2+\omega^2)$ or $\omega \operatorname{Im}[S(\sigma+i\omega)]/(\sigma^2+\omega^2)$ may possibly yield delta functions or their derivatives in the limit $\sigma\to 0$. Nevertheless, whenever the rationalization leading to the forms (17) and (18) is allowable, we have again failed to discover any reasonable cases for which it is invalid first to carry out the limit $\sigma\to 0$ for $S(\sigma+i\omega)$ separately in these equations, then to complete the limiting process for the remaining functions of σ. This procedure is equivalent to using the relation $Q(i\omega)=\lim_{\sigma\to 0}[S(i\omega)/(\sigma+i\omega)]$ instead of the correct form (14). The result is

$$J(\omega)=\omega^{-1}T(\omega)+\pi\delta(\omega)P(\omega), \qquad (17')$$

$$H(\omega)=\omega^{-1}P(\omega)-\pi\delta(\omega)T(\omega). \qquad (18')$$

These results hold only when the rationalization leading to (17) and (18) is allowable.[40] They show that it is not correct in general to divide through (15′) and (16′) by ω to obtain the desired relations; instead, $\delta(\omega)$ terms must be added to account properly for the singular case $\omega=0$. Again, we suggest that as long as $S(p)$ is an analytic function of p, (17′) and (18′) will always represent a valid simplification of (17) and (18). As an example, we may consider $S(p)=p$, which is analytic except at $p=\infty$. Then, (18) yields zero identically and so does (18′) when it is noted that $P(\omega)=0$ and $T(\omega)=\omega$. The results (17′) and (18′) also hold for some functions such as $S(p)=(p-1)^{-1}$ which are not analytic in the entire finite part of the right half of the p plane. On the other hand, they fail for a nonanalytic function such as $S(p)=p^*=\sigma-i\omega$.

Finally, we may replace $P(\omega)$ in (17′) by $P(0)$ except when $P(\omega)$ is an impulse function (see end of Appendix II). As we shall see later, $P(\omega)$ is an even function of ω and $T(\omega)$ an odd function. Therefore, $P(\omega)$ may be expanded for small ω in a series which may contain a constant term, $P(0)$, plus powers of ω^2. Since negative powers are physically unrealizable, the term $\pi\delta(\omega)P(\omega)$ in (17′) must be finite or an impulse function but may be zero. On the other hand, when $T(\omega)$ is expanded in a series around $\omega=0$, it is found that the series can contain only terms in ω^{2n-1}, where $n=0, 1, 2, 3, \cdots$. Only the term containing ω^{-1} can yield a nonzero value in $\pi\delta(\omega)T(\omega)$ and the nonzero value obtained when ω^{-1} is actually present in the series is proportional to $\omega^{-1}\delta(\omega)$, a quantity discussed in Appendix II. We may now rewrite the final simplified forms of (17) and (18) as

$$J(\omega)=\omega^{-1}T(\omega)+\pi\delta(\omega)P(0), \tag{17''}$$

$$H(\omega)=\omega^{-1}P(\omega)-\pi\delta(\omega)\{\omega^{-1}[\omega T(\omega)]_{\omega=0}\}. \tag{18''}$$

These relations are applicable so long as $P(\omega)$ and $T(\omega)$ are not impulse functions and the rationalization leading to (17) and (18) is allowable.[40]

Next, let us separate Eqs. (8) and (9) into real and imaginary parts. We find[42]

$$P(\omega)=\lim_{\sigma\to 0}\int_0^\infty B(t)e^{-\sigma t}\cos\omega t dt\equiv\mathfrak{F}_c[B(t)], \tag{21}$$

$$T(\omega)=-\lim_{\sigma\to 0}\int_0^\infty B(t)e^{-\sigma t}\sin\omega t dt\equiv-\mathfrak{F}_s[B(t)], \tag{22}$$

$$J(\omega)=\lim_{\sigma\to 0}\int_0^\infty A(t)e^{-\sigma t}\cos\omega t dt\equiv\mathfrak{F}_c[A(t)], \tag{23}$$

and

$$H(\omega)=\lim_{\sigma\to 0}\int_0^\infty A(t)e^{-\sigma t}\sin\omega t dt\equiv\mathfrak{F}_s[A(t)]. \tag{24}$$

We have not actually carried out the limit $\sigma\to 0$ in the above equations because the exponential factor may be required for convergence in certain cases. In such cases, σ may be set to zero after integration. The above equations are generalized Fourier sine and cosine transforms,[3] which we denote symbolically with the operators $\mathfrak{F}_s$ and $\mathfrak{F}_c$. These results underline the importance of the $B(t)$ and $A(t)$ functions, since they show how simple, well-tabulated integral transforms may be employed to obtain the four real quantities $P(\omega)$, $T(\omega)$, $J(\omega)$, and $H(\omega)$ from $B(t)$ and $A(t)$. Note that the above transforms may also be considered as real Laplace transforms with σ taking the place of p. A simple case where the convergence factors are necessary is given by $B(t)=u_0(t)$; for this choice of $B(t)$, $S(p)=p^{-1}$ and $S(i\omega)=\pi\delta(\omega)-i\omega^{-1}$. If $S(i\omega)$ is interpreted as an input admittance, the system is an ideal inductance.[43] It should be mentioned that if $B(t)$ or $A(t)$ contain $\delta(t)$, the range of integration in the above equations must be extended slightly negative to cover the region around $t=0$. We have hitherto indicated such extension by writing the lower limit as $0-$; from now on, we shall omit this refinement and let it be understood when necessary. It is usually then convenient to take $\sigma=0$ and to extend the lower limit to $-\infty$; this procedure is allowed since $B(t)$ and $A(t)$ may be taken identically zero for any finite nonzero negative t. We shall assume that this extension will be made when necessary.

Equations (21) through (24) may be inverted by the methods of Appendix III to yield

$$B(t)=J(\infty)\delta'(t)+(2/\pi)\mathfrak{F}_c[P(\omega)], \tag{25}$$

$$B(t)=P(\infty)\delta(t)-(2/\pi)\mathfrak{F}_s[T(\omega)], \tag{26}$$

$$A(t)=(2/\pi)\mathfrak{F}_c[J(\omega)], \tag{27}$$

$$A(t)=J(\infty)\delta(t)+(2/\pi)\mathfrak{F}_s[H(\omega)]. \tag{28}$$

If the above functions of ω involve impulse functions at the origin such as $\delta(\omega)$, σ may be taken zero in the convergence factors, the lower limits of integration may be extended to $-\infty$, and the factor $2/\pi$ replaced by $1/\pi$. This procedure is valid because all the integrands are even functions of ω. This and earlier statements concerning the evenness and oddness of the above functions of ω are easily verified from Eqs. (21) through (24). Thus, for example, $J(\omega)$ can only be even since ω is involved on the right of (23) only in the even function $\cos\omega t$.

The above results show that the impulse and step-function responses can be obtained from the real and imaginary parts of $S(i\omega)$ and $Q(i\omega)$. If $J(\infty)$ and $P(\infty)$ are nonzero, however, it is clear that $A(t)$ and $B(t)$ are only fully determined by $J(\omega)$, not by any of the other three quantities. Thus, if $J(\omega)$, which may typically be the real part of a complex or anomalous dielectric constant, is known for all frequencies, then the indicial response $A(t)$ is fully determined. This quantity, in turn, determines $Q(p)$ and $S(p)$ through (9) and (7) so that $J(\omega)$ contains complete information about the system and specifies it completely.

It is next of interest to obtain direct transform relations between $P(\omega)$ and $T(\omega)$ and between $J(\omega)$ and $H(\omega)$. Substituting (26) in (21) yields

$$P(\omega)=P(\infty)-(2/\pi)\mathfrak{F}_c\mathfrak{F}_s[T(y)]. \tag{29}$$

Similarly, (25) and (22) give

$$T(\omega)=\omega J(\infty)-(2/\pi)\mathfrak{F}_s\mathfrak{F}_c[P(y)]. \tag{30}$$

We also readily find the additional relations

$$J(\omega)=J(\infty)+(2/\pi)\mathfrak{F}_c\mathfrak{F}_s[H(y)], \tag{31}$$

$$H(\omega)=(2/\pi)\mathfrak{F}_s\mathfrak{F}_c[J(y)]. \tag{32}$$

Although the above relations are written in terms of generalized Fourier sine and cosine transforms, the convergence factors will usually be unnecessary and the transforms will then degenerate into ordinary sine and cosine transforms of which extensive tables are available.[1,2] Even when the convergence factors are necessary, the desired integrals may often be found tabulated directly[44] or may be obtained from Laplace transform tables. These tables, together with the properties of delta functions, make it usually a simple matter to evaluate any of the previous $\mathfrak{F}_c$ and $\mathfrak{F}_s$ transforms in practical cases. It should be pointed out that connections of the form of (29) and (30) without the $P(\infty)$ and $J(\infty)$ terms and expressed as ordinary Fourier sine and cosine transforms have been mentioned by Titchmarsh.[45]

The above relations may be transformed, as shown in Appendix IV, into the more familiar forms

$$P(\omega)=P(\infty)-\frac{2}{\pi}⨍_0^\infty \frac{yT(y)dy}{y^2-\omega^2}, \tag{29'}$$

$$T(\omega)=\omega J(\infty)-\frac{2\omega}{\pi}⨍_0^\infty \frac{P(y)dy}{\omega^2-y^2}, \tag{30'}$$

$$J(\omega)=J(\infty)+\frac{2}{\pi}⨍_0^\infty \frac{yH(y)dy}{y^2-\omega^2}, \tag{31'}$$

$$H(\omega)=\frac{2\omega}{\pi}⨍_0^\infty \frac{J(y)dy}{\omega^2-y^2}. \tag{32'}$$

These are just the Kronig-Kramers transform relations.[20,21,46] Their expression in (29) through (32) as double (generalized) Fourier transforms shows that they are somewhat analogous to the Stieltjes transform which is an iterated Laplace transform.[47]

The bar through the integral in the above equations indicates that the principal value is to be understood. The Kronig-Kramers relations are of great importance in circuit-theory[48] and in magnetic and dielectric susceptibility measurements. Aside from the limiting values at infinite frequency, which are often zero, they show that measurements over all frequencies of importance of one component, such as the real part of an impedance, allow the value of the corresponding component to be calculated for any frequency. The relations are thus useful whenever it is easier or more convenient to measure one component of $S(i\omega)$ or $Q(i\omega)$ than the other. Note that (29′) and (31′) become particularly simple when the values of $P(\omega)$ and $J(\omega)$ at zero frequency are required and ω is taken zero. Some mathematical considerations pertaining to the Kronig-Kramers relations have been given by Gross.[49]

There remain a few more general relations of some interest. Substitution of (5) into (21) and (22) yields

$$P(\omega)=A(0)+\mathfrak{F}_c\left[\frac{dA(t)}{dt}\right], \tag{33}$$

$$T(\omega)=-\mathfrak{F}_s\left[\frac{dA(t)}{dt}\right]. \tag{34}$$

A useful relation may also be produced by substituting (17′) and (18′) in (32′), then multiplying both sides of the equation by ω. The result is

$$P(\omega)=P(0)+\frac{2\omega^2}{\pi}⨍_0^\infty \frac{T(y)dy}{y(\omega^2-y^2)}, \tag{35}$$

which has been derived previously by Bode.[50]

Other interesting relations may be formed, e.g., by multiplying (21) by $\mathfrak{F}_s$ and (22) by $\mathfrak{F}_c$. The results are

$$\mathfrak{F}_s[P(\omega)]=\mathfrak{F}_s\mathfrak{F}_c[B(t)], \tag{36}$$

$$\mathfrak{F}_c[T(\omega)]=-\mathfrak{F}_c\mathfrak{F}_s[B(t)]. \tag{37}$$

Now, as shown in Appendix IV, the right-hand sides are equivalent to the Kronig-Kramers integrals. We have thus shown that these ubiquitous relations apply between functions of time as well as of frequency. They are not conjugate in the time domain as they are in the frequency domain, however.

Finally, it is desirable to obtain direct transform relations between $S(p)$ and $Q(p)$ and the real and imaginary parts of $S(i\omega)$ and $Q(i\omega)$. Such relations may be obtained by substituting (25) and (26) into (8) and (27) and (28) into (9). Such substitution yields equations involving the product operators $\mathfrak{L}\mathfrak{F}_c$ and $\mathfrak{L}\mathfrak{F}_s$. These products may be simplified by changing the order of integration and carrying out one integration. We obtain the results

$$S(p)=pJ(\infty)+\frac{2p}{\pi}\int_0^\infty \frac{P(y)dy}{p^2+y^2}, \tag{38}$$

$$S(p)=P(\infty)-\frac{2}{\pi}\int_0^\infty \frac{yT(y)dy}{p^2+y^2}, \tag{39}$$

$$Q(p)=\frac{2p}{\pi}\int_0^\infty \frac{J(y)dy}{p^2+y^2}, \tag{40}$$

$$Q(p)=J(\infty)+\frac{2}{\pi}\int_0^\infty \frac{yH(y)dy}{p^2+y^2}, \tag{41}$$

where the convergence factors have been omitted as unnecessary. Since all the integrands are even functions, the integrals may be divided by two and limits of $-\infty$ to ∞ taken when desired. Although these integrals have a superficial resemblance to the Kronig-Kramers relations, they are not principal values. If, however, p is taken as $\sigma+i\omega$ in the above equations, and the limit $\sigma\to 0$ carried out, the right side of each reduces to $S(i\omega)$ or $Q(i\omega)$ as the case may be with either the real or imaginary term expressed by means of a Kronig-Kramers transform. The expression (38) is an alternative form of a relation given first by Cauer[51,52] who expressed it as a Stieltjes integral. Somewhat less general and complete forms of (38) and (39)[11] and of (41)[43] have been given by Gross. It is not surprising that Eqs. (38) to (41) exist, since we have already shown that a real function such as $J(\omega)$ may be transformed to yield $A(t)$, and $A(t)$ then transformed to $Q(p)$. The above equations simply represent the elision of two transforms into a single transform. They are obviously useful in showing how $S(p)$ or $Q(p)$ can be calculated from knowledge of one of the real functions on the right plus the value of $J(\omega)$ or $P(\omega)$ at infinity.

The integral transform relations between $A(t)$ and $B(t)$ and Q and S are often presented[1,14,22,42,43] in the form of exponential Fourier transforms such as

$$S(i\omega)=\int_{-\infty}^{\infty} B(t)e^{-i\omega t}dt\equiv\mathfrak{F}_e[B(t)], \qquad (42)$$

and

$$Q(i\omega)=\int_{-\infty}^{\infty} A(t)e^{-i\omega t}dt\equiv\mathfrak{F}_e[A(t)], \qquad (43)$$

where the symbol $\mathfrak{F}_e$ denotes the exponential Fourier transform. The above relations may be obtained from (8) and (9) by taking $\sigma=0$ and rewriting these equations as bilateral transforms, which is, of course, permitted since $A(-t)=B(-t)=0$. Most of our previous results can be obtained from (42) and (43), but they will not then include the convergence factors such as $\exp(-\sigma t)$ which arose naturally from our present starting point. The lack of convergence factors makes extreme care necessary in the interpretation of such equations as (42) and (43) when the integrands contain functions of $\delta(t)$ or $u_0(t)$. The integrals may not then exist in the classical Riemann-Lebesgue sense. These difficulties may be avoided by interpreting the integrals in the sense of distribution theory,[53–56] which is equivalent to a mathematically consistent extension of the meaning of the integrals beyond those of Riemann and Lebesgue. Such complications are, however, automatically avoided in the present treatment and, although distribution theory promises to be of considerable usefulness in linear-system theory, we shall not consider it further herein except insofar as some of its results are used in Appendix II.

It should be noted that when they converge (42) and (43) are often very useful because they yield $S(i\omega)$ and $Q(i\omega)$ directly by means of a well-known, tabulated integral transform. On the other hand, the Laplace transform relations (8) and (9), while they are more convergent, yield $S(p)$ and $Q(p)$. Although it is always possible to obtain $S(i\omega)$ and $Q(i\omega)$ from $S(p)$ and $Q(p)$ by separation into real and imaginary parts and by then taking the limit $\sigma\rightarrow 0$ properly, it is often convenient to obtain these quantities directly in a single transform step. It is, of course, just when the exponential Fourier integrals do not converge properly that care must be taken in carrying out the limit $\sigma\rightarrow 0$ in passing from $S(p)$ and $Q(p)$ to $S(i\omega)$ and $Q(i\omega)$.

The interrelations between the quantities characterizing the linear system are presented in pictorial form in Fig. 1. Lines with arrows on both ends indicate a reciprocal relationship. Knowledge of either function connected by such lines allows the other to be uniquely determined in one step. Lines with only a single arrow indicate that the function at the arrow end of the line may be determined directly from knowledge of the function at the other end. The numbers in parentheses are the equation numbers specifying the given trans-

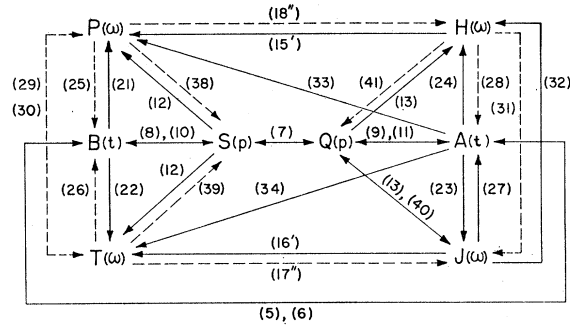

FIG. 1. Diagram showing the connections between the various quantities characterizing a linear system.

form. Finally, dotted lines indicate that the function at the arrow end can only be determined within a constant (which may be zero) from the function at the other end. The constant is, of course, not arbitrary, but is fully determined by the system considered. It is worthwhile pointing out that there are no solid paths leading away from any function where a dotted path originates.[57] There thus exists no multiple sequence of operations whereby such a "weak" function alone can in general yield full knowledge of the complete system. Note that of the four functions $P(\omega)$, $T(\omega)$, $H(\omega)$, and $J(\omega)$, only $J(\omega)$ is not weak in the above sense. It is the only one of the four which contains complete information about the system.

A simple example with which to illustrate some of the foregoing results is given by a series RC circuit. If the system function is taken as the input admittance, it is

$$S(p)=\frac{1}{R}\left[\frac{p}{\alpha+p}\right]=\frac{1}{R}\left[1-\frac{\alpha}{\alpha+p}\right], \qquad (44)$$

where $\alpha=(RC)^{-1}$. Upon expansion of (44) into real and imaginary parts, one finds on then letting $\sigma\rightarrow 0$ that

$$P(\omega)=\frac{1}{R}\frac{\omega^2}{\alpha^2+\omega^2}, \qquad (45)$$

$$T(\omega)=\frac{\alpha}{R}\frac{\omega}{\alpha^2+\omega^2}. \qquad (46)$$

The inverse Laplace transform of $S(p)$ is

$$B(t)=\frac{1}{R}\delta(t)-\frac{\alpha}{R}e^{-\alpha t}. \qquad (47)$$

Equation (6) now yields

$$A(t)=\frac{1}{R}e^{-\alpha t}. \qquad (48)$$

The same result could have been obtained directly by transforming $Q(p)=[R(\alpha+p)]^{-1}$. It is now easily verified that (21) and (22) yield $P(\omega)$ and $T(\omega)$ as given by (45) and (46). Similarly, (23) and (24) immediately yield the correct values

$$J(\omega)=\frac{\alpha}{R}\frac{1}{\alpha^2+\omega^2}, \tag{49}$$

and

$$H(\omega)=\frac{1}{R}\frac{\omega}{\alpha^2+\omega^2}, \tag{50}$$

which are just Debye dispersion equations with the time constant $\tau_0=RC=\alpha^{-1}$. It is also easy to show that all the transform relations between $A(t)$ and $B(t)$ and the functions of frequency hold.

The Debye dispersion equations are of great importance in dielectric[15,58] and magnetic studies[59] both because of their simplicity and their generality. They appear in systems which contain only one kind of storage element, such as capacitance, inductance, or mass, together with dissipation. When the equilibrium condition of such a system is disturbed, it decays to the original or a new equilibrium state exponentially with a single time constant as shown in (48). In the next section, we shall discuss systems exhibiting many relaxation time constants and continuous distributions of such time constants. When two different energy storage mechanisms, such as both inductance and capacitance, are simultaneously present in a system, it may exhibit resonance and a definite resonant frequency; then, the Debye equations will no longer be applicable. This case of resonant behavior is discussed in the final section.

It is next of interest to allow the capacitance C to become infinite. Thus, $\alpha\to 0$, and the system degenerates to a resistance R alone. Then $S(p)=1/R$ and $Q(p)=1/Rp$. On carrying out the limit $\alpha\to 0$ for the above results or calculating directly from these values of $S(p)$ and $Q(p)$, we obtain

$$P(\omega)=R^{-1}, \tag{51}$$

$$T(\omega)=R^{-1}\pi\omega\delta(\omega)=0, \tag{52}$$

$$B(t)=R^{-1}\delta(t), \tag{53}$$

$$J(\omega)=R^{-1}\pi\delta(\omega), \tag{54}$$

$$H(\omega)=R^{-1}\omega^{-1}, \tag{55}$$

$$A(t)=R^{-1}u_0(t). \tag{56}$$

We have, therefore, obtained the surprising result that a pure resistance yields nonzero real and imaginary parts for $Q(i\omega)$. Thus, just as an ideal reactance involves a delta function as its Kronig-Kramers conjugate pair, an ideal resistance also leads to a delta function at the $Q(i\omega)$ level. It is interesting to observe that it is the second term only in (17′) which contributes to $J(\omega)$ when this quantity is calculated from $T(\omega)$ and $P(\omega)$. Note that both Kronig-Kramers relations hold for the $J(\omega)$ and $H(\omega)$ of (54) and (55). All of the above results could, of course, have been obtained by taking $S(p)$ as the impedance of a parallel RL circuit and allowing L to go to zero.

The conditions of physical realizability of a linear system are well known[52,60,61]; we shall merely summarize them here. First, it is obviously necessary that the temporal response of the system to excitation, e.g., $A(t)$ or $B(t)$, must be real and not complex. Further, as discussed earlier, the response must be zero for $t<0$ if the system is initially quiescent. If Eq. (10) or (11) is evaluated by complex integration with $\sigma=0$, for $t<0$ the path of integration may be closed with a large semicircle in the right-half of the complex p plane. For the integral to be zero, as required by the fact that there can be no response before excitation, it is necessary that the sum of the residues of poles within the semicircle be zero. If this condition is met for a nonzero number of poles in this region, these poles will lead to transient response which grows indefinitely with increasing time rather than to "constant" or decaying transients. Such behavior is only possible (for a limited time) with an active system, which can itself supply power to the output irrespective of the input source of power.[62] The system is thus unstable. For a passive system, it is therefore necessary that there be no poles in this region, (with the possible exception of the points $p=0$, $p=\infty$; see later discussion) and $S(p)$ is then analytic there. Thus, for a passive system with loss, the transient response must eventually approach zero at sufficiently long times. In this case, $B(\infty)=0$ and $A(\infty)\equiv P(0)=0$. For lumped constant systems, $S(p)$ is also a rational algebraic function of p. It is then termed a positive real function.[52,61] Such a function satisfies the conditions that $S(p)$ is real when p is real and that $\mathrm{Re}[S(p)]\geqslant 0$ when $\sigma\geqslant 0$. This last condition ensures that there are no negative resistances in the system, since their presence would render the system active. For such a positive-real $S(p)$, the highest powers of p in the numerator and denominator of $S(p)$ can differ by unity at most. The poles and zeros of $S(p)$ are restricted to the $i\omega$ axis of the p plane and to its left-half. Further, any nonreal poles or zeros must occur in conjugate complex pairs, and any poles on the real axis must be simple with positive-real residues.

If $S(p)$ is a positive-real function, the system must satisfy an ordinary linear differential equation with constant coefficients. The coefficients must themselves be positive-real. Note that the condition of analyticity of $S(p)$ requires that the Cauchy-Riemann conditions hold between its real and imaginary parts ($\sigma\neq 0$).[63] Then, the real and imaginary parts with $\sigma\to 0$ are also connected by Hilbert transforms.[64,65] In the special case of even real part and odd imaginary part, the Hilbert transforms immediately simplify to the Kronig-Kramers relations.[66] When $B(t)$ and $A(t)$ are zero for $t<0$ as required by the causality condition, Eqs. (21) through (24) show that the real and imaginary parts of $S(i\omega)$ and $Q(i\omega)$ will indeed be even and odd in ω as stated earlier. Thus, for an analytic $S(p)$, the real and imaginary parts of $S(i\omega)$ will obey the Kronig-Kramers relations. The converse statement is not necessarily implied; in the present work and elsewhere[31] we have derived such relations between the real and imaginary parts by methods which do not explicitly require that $S(p)$ be analytic in the entire right half plane. Note that even if $S(p)$ is a positive real function, $Q(p)$ need not be. This conclusion is immediately evident from the fact that if $S(p)$ is a positive real function, the highest powers of p in the numerator and denominator of $Q(p)$ can differ by zero, minus one, and minus two at most. Elsewhere, we have shown that if $Q(p)$ is a rational function of p having unity numerator and a denominator containing a sum of terms with finite, integral, positive powers of p, then $Q(p)$ satisfies the Kronig-Kramers relations.[32]

The foregoing integral relations are, of course, of far wider scope than is indicated by their usual restriction to rational positive real functions. For example, they also apply to irrational functions such as $S(p)=[p+a]^{-\frac{1}{2}}$. The impulse response for this system function is readily found from the transform tables to be $B(t)=[4\pi^{-1}t]^{\frac{1}{2}}\exp(-at)$. The equations also apply to meromorphic functions, to entire functions, and to multiple-valued functions such as $S(p)=\tan^{-1}(ap^{-1})=\pi/2-\tan^{-1}(p/a)$. This last is an interesting choice; the corresponding impulse response is $B(t)=\sin at/t$, from which we readily obtain $P(\omega)=(\pi/2)[u_0(\omega)-u_0(\omega-a)]$ and $T(\omega)=(1/2)\ln|(\omega-a)/(\omega+a)|$. Here, the frequency response of the real part is that of an ideal resistor until $\omega=a$; thereafter, it is zero. It may be pointed out that when $S(p)$ is not a simple rational (or meromorphic) function of p, the system need not be made up of lumped elements, and it will in general, satisfy a linear partial differential equation or several such equations, as in the case of a transmission line.

III. DISTRIBUTION FUNCTION RELATIONS

Dielectric constant measurements are often analyzed in terms of a distribution of relaxation times.[16,67,68] It is assumed that when the observed results cannot be explained in terms of simple Debye dispersion, which involves only a single relaxation time, they instead arise from processes which lead to an entire line spectrum or continuous spectrum of relaxation times. Such a connection between observed results and a distribution of relaxation times is of considerable generality and will be considered in detail in the present section. Even though it was originally introduced to treat nonresonant absorption of the Debye type, it is formally applicable as well to resonance measurements, as we shall show later.

It should be noted, however, that the results of this section are less general than those of the previous one. There, all equations applied to the general linear system. Here, we are only concerned with relaxation systems; that is, linear systems whose response to a step function input monotonically decreases, or relaxes, towards zero. Examples of such systems are RL and RC circuits and over-damped RLC circuits.

There are several equations any one of which might be taken as the fundamental relation between experimental results and the distribution-of-relaxation-times function $G(\tau)$. For example, it is quite possible to start the analysis with a relation between the step function response of the system $A(t)$ and an integral transform of $G(\tau)$. Such an equation can be justified directly from physical reasoning.[67] We shall, however, begin with the following equation from which the above relation will later be derived

$$Q(p)=\int_0^\infty \frac{G(\tau)d\tau}{1+p\tau}. \tag{57}$$

The physical meaning of this equation is quite clear. The network function $p^{-1}Y(p)=Q(p)$ for simple Debye dispersion is $[1+p\tau_0]^{-1}$; in (57) such a function is averaged over a distribution $G(\tau)$ of relaxation times.

It should be emphasized that Eq. (57) is written for a network function $Q(p)$ which satisfies the condition $Q(\infty)\equiv J(\infty)=0$. Such a definition allows us to eliminate $J(\infty)$ terms which would otherwise appear in equations such as (57). When $J(\infty)\neq 0$, it is a simple matter to redefine a new $Q(p)$ for which $Q(\infty)=0$ by subtracting $J(\infty)$ from the original $Q(p)$.

In order that the distribution function $G(\tau)$ be physically reasonable and realizable, it must satisfy certain conditions. First, it must be real and always positive for any real value of the real variable τ between 0 and ∞. It is also physically unreasonable to allow it to have poles within this span although it may contain delta functions if there are isolated lines in the spectrum of distribution times. Finally, it must, from (57), be normalized so that

$$\int_0^\infty G(\tau)d\tau=Q(0)=J(0). \tag{58}$$

We shall usually take $J(0)=1$. If any given $Q(p)$ has a $J(0)$ different from unity, it may be normalized by dividing by $J(0)$ unless this quantity is zero, infinite, or an impulse function at the origin. When $Q(0)$ does not exist, Eq. (58) is meaningless but (57) will still hold over the rest of the range of p provided the choice of $G(\tau)$ is such as to make the integral convergent exclusive of the point $p=0$. We shall usually assume that such normalization has been carried out. It is pertinent to mention, however, that from a purely mathematical viewpoint many of the above conditions on $G(\tau)$ are too restrictive. For example, it may be negative over part of its range, and it need not be normalized nor normalizable, provided only that the integral (57) is convergent.

Equation (57) is more general than that of most previous work. Usually,[67,69] an analogous relation is written in terms of $i\omega$ instead of p. The integral over $G(\tau)$ then yields $Q(i\omega)$ instead of $Q(p)$. Although it is always relatively easy to pass directly from $Q(p)$ to $Q(i\omega)$, it is sometimes more difficult to go in the reverse direction, from knowledge of $Q(i\omega)$ to that of $Q(p)$. This transition may, of course, be carried out by means of the integral relations between $J(\omega)$, $A(t)$, and $Q(p)$ presented in the last section, but these transforms may be difficult, if not impossible, to evaluate in certain cases. It is therefore valuable to begin with a single transform relation between $G(\tau)$ and $Q(p)$. Whenever $Q(i\omega)$ rather than $Q(p)$ is desired from $G(\tau)$, (57) may be written in the more conventional form

$$Q(i\omega)=\int_0^\infty \frac{G(\tau)d\tau}{1+i\omega\tau}. \tag{59}$$

This equation yields the well-known additional relations

$$J(\omega)=\int_0^\infty \frac{G(\tau)d\tau}{1+(\omega\tau)^2}, \tag{60}$$

and

$$H(\omega)=\int_0^\infty \frac{(\omega\tau)G(\tau)d\tau}{1+(\omega\tau)^2}. \tag{61}$$

Equation (57) may be easily transformed to several more useful and familiar forms. In (57), let $\lambda\equiv\tau^{-1}$ and $\lambda^{-1}G(\lambda^{-1})\equiv D(\lambda)$, a new distribution function. Then (57) becomes

$$Q(p)=\int_0^\infty \frac{D(\lambda)d\lambda}{p+\lambda}\equiv\mathcal{L}\mathcal{L}[D(\lambda)]\equiv\mathcal{S}[D(\lambda)]. \tag{57$'$}$$

The symbol $\mathcal{S}$ denotes the Stieltjes transform[47,70,71]; it is an iterated Laplace transform as indicated. We may also write from (57)

$$S(p)=pQ(p)=p\mathcal{S}[D(\lambda)]. \tag{62}$$

These are useful relations since tables of Stieltjes transforms exist.[2]

Another method of transforming Eq. (57) is to divide numerator and denominator by p. Then, we obtain

$$Q(p)=\frac{1}{p}\int_0^\infty \frac{G(\tau)d\tau}{p^{-1}+\tau}=p^{-1}\mathcal{S}_{(1/p)}[G(\tau)], \tag{57$''$}$$

and

$$S(p)=\mathcal{S}_{(1/p)}[G(\tau)], \tag{62$'$}$$

where now $\mathcal{S}_{(1/p)}$ indicates that the independent variable of the Stieltjes transform is p^{-1}.

Now, on substituting (9) for $Q(p)$ in (57′) and multiplying both sides of the equation by $\mathcal{L}^{-1}$, the inverse Laplace transform operator, we obtain

$$A(t)=\mathcal{L}[D(\lambda)]=\int_0^\infty G\left(\frac{1}{\lambda}\right)e^{-t\lambda}\frac{d\lambda}{\lambda}. \tag{63}$$

We have thus easily obtained the relation mentioned earlier between $G(\tau)$ and the step function response. Alternatively, this equation could have been used to derive (57). An equation relating $B(t)$ and a transform of $G(\tau)$ may be derived in a manner similar to that leading to (63); such a relation is unnecessary, however, since once $A(t)$ is known, $B(t)$ may be obtained from (5).

When the inverse transformation exists, we may also write from (63)

$$D(\lambda)=\mathcal{L}^{-1}[A(t)]=\lambda^{-1}G(\lambda^{-1}). \tag{64}$$

We may then obtain $G(\tau)$ from

$$G(\tau)=\tau^{-1}D(\tau^{-1}). \tag{64$'$}$$

The above results, therefore, enable us to obtain $G(\tau)$ if the inverse Laplace transform of $A(t)$ can be found. We thus see that $A(t)$ occupies an intermediate position between the distribution function $D(\lambda)$, its inverse Laplace transform, and the network function $Q(p)$, its direct Laplace transform.

Equation (63) may be inverted less formally than in (64) by using Fourier's integral formula, as shown by Titchmarsh.[72] The result may be written for our case as

$$G(\tau)=\frac{1}{2\pi\tau}\int_{-\infty}^\infty A(iy)e^{iy/\tau}dy \quad (\tau>0), \tag{65}$$

an exponential Fourier transform. As Titchmarsh points out, such an equation can only hold when $A(t+iy)$ is an analytic function regular for $t>0$. The above result therefore depends upon the possibility of analytic continuation of $A(t)$ in the complex plane. Equation (65) was first used in connection with relaxation distribution functions by Simha.[9]

Next, it will be of interest to consider the direct inversion of the above Stieltjes transforms. Later, we shall show how inversion may sometimes be accomplished by transforming the pertinent quantities to the Mellin transform plane, then using relations which hold there, and finally transforming the desired quantity back to the domain of the original variables by means of an inverse Mellin transform. Often, however, this final transform is difficult or impossible to carry out. Titchmarsh[73] has given a result, however, which is often easy to use. Applied to (57′) it yields,

$$D(\lambda)=\frac{i}{2\pi}[Q(\lambda e^{i\pi})-Q(\lambda e^{-i\pi})]. \tag{66}$$

This equation also depends on the feasibility of analytic continuation of $Q(p)$ and hence on its analytic character. When $D(\lambda)$ is a delta function of λ or a sum of such functions, a limiting process must be carried out. First, Eq. (66) may be simplified by recognizing that the two functions on the right are conjugate. We obtain

$$\begin{aligned} D(\lambda)&=\pi^{-1}\,\mathrm{Im}[Q(\lambda e^{-i\pi})] \\ &=\lim_{\epsilon\to 0}\pi^{-1}\,\mathrm{Im}[Q(-\lambda-i\epsilon)], \end{aligned} \tag{66$'$}$$

where the second form shows explicitly the limiting process represented symbolically in the preceding equation. A result equivalent to (66′) was first given by Gross in connection with viscoelasticity theory.[10]

An example of a case to which (66) or (66′) may be applied without limiting is given by

$$Q(p)=[1+(p\tau_0)^2]^{-1}[1+\pi^{-1}p\tau_0\ln(p\tau_0)^2]. \quad (67)$$

For later reference, we may write the real and imaginary parts of the corresponding $Q(i\omega)$ function; these are

$$J(\omega)=[1+|\omega\tau_0|]^{-1}, \quad (68)$$

and

$$H(\omega)=\frac{\omega\tau_0\ln(\omega\tau_0)^2}{\pi[(\omega\tau_0)^2-1]}. \quad (69)$$

Since the logarithm in (67) is multiple-valued, separation of $Q(p)$ into real and imaginary parts in the limit $\sigma\rightarrow 0$ is not unique, and the sign of $(\omega\tau_0)$ in (68) may be either plus or minus. However, for $J(\omega)$ to represent a realizable system, it is necessary that it be even in ω as shown. A functional dependence of the form (68) is often found in dielectric constant measurements when there is a wide distribution of relaxation times such as is often found in a distributed system. When (66) is applied to (67) we obtain

$$D(\lambda)=\frac{2\lambda\tau_0}{\pi[1+(\lambda\tau_0)^2]}. \quad (70)$$

From (64′), $G(\tau)$ is then

$$G(\tau)=(2/\pi\tau_0)[1+(\tau/\tau_0)^2]^{-1}, \quad (71)$$

a continuous distribution. Since $Q(0)=J(0)=1$, $G(\tau)$ is normalized to unity. Finally, since in the present case (70) is listed in the table of Stieltjes transforms,[2] it may be readily verified that (67) is the transform of (70) as specified by (57′).

Equation (71) represents too wide a distribution of relaxation times to be physically realizable. It will be noted that it specifies that there are some relaxation times present having any arbitrary finite value. Physically, however, there is a short relaxation time limit and no relaxation times shorter than the limiting value can occur. To make (71) physically realizable, it is necessary to take $G(\tau)=0$ for $\tau<\tau_{\min}$. Since $\tau_{\min}$, the shortest possible relaxation time, may be much shorter than τ_0, such a limitation may not affect the frequency response as exemplified by (68) and (69) until $(\omega\tau_0)\gg 1$. Thus, although one may measure a $J(\omega)$ dependence of the form of (68) over a wide range of ω, there must finally be a deviation from such response at sufficiently high frequencies. This deviation will take the form of a final more rapid decrease of $J(\omega)$ with increasing ω than that specified by (68).

It is worth emphasizing that when a $G(\tau)$ (continuous or discontinuous) is used to calculate $J(\omega)$ or $H(\omega)$ from (60) or (61), the resulting functions must be correctly of even and odd parity respectively in ω. This result follows from the fact that $G(\tau)$ does not contain ω and that (60) involves it only as ω^2 and (61) as $\omega f(\omega^2)$. The fact that any $G(\tau)$ function will lead to $J(\omega)$ and $H(\omega)$ functions of proper parity does not ensure, however, that these functions belong to a $Q(p)$ which is an analytic function of p in the right half p plane or that the causality condition holds. An example is the function of (67); it will only be analytic if an infinite system of Riemann sheets and cuts are used to make it single-valued.

We may also apply Titchmarsh's result to (62′) to yield

$$G(\tau)=\frac{i}{2\pi}[S(\tau^{-1}e^{-i\pi})-S(\tau^{-1}e^{i\pi})],$$

or

$$\begin{aligned}G(\lambda^{-1})&=-\pi^{-1}\,\mathrm{Im}[S(\lambda e^{-i\pi})]\\&=-\lim_{\epsilon\rightarrow 0}\pi^{-1}\,\mathrm{Im}[S(-\lambda-i\epsilon)].\end{aligned} \quad (72)$$

With $S(p)=pQ(p)$, (67) and (72) again give (71).

If we use Eqs. (59), (60), and (61) as definitions of $Q(i\omega)$, $J(\omega)$, and $H(\omega)$ in terms of $G(\tau)$, we may derive a number of additional interesting relations by doubly transforming $G(\tau)$ in various ways. Some results thus obtained are

$$\omega Q(i\omega)=\mathfrak{L}_{(1/\omega)}\mathfrak{F}_e[G(\tau)], \quad (73)$$

$$\omega J(\omega)=\mathfrak{L}_{(1/\omega)}\mathfrak{F}_c[G(\tau)]=\mathfrak{F}_{s(1/\omega)}\mathfrak{L}[G(\tau)], \quad (74)$$

$$\omega H(\omega)=\mathfrak{L}_{(1/\omega)}\mathfrak{F}_s[G(\tau)]=\mathfrak{F}_{c(1/\omega)}\mathfrak{L}[G(\tau)]. \quad (75)$$

Equation (73) only holds so long as $G(-\tau)$ is taken zero, a physically reasonable assumption. In the above equations, $\mathfrak{F}_e$ is the exponential Fourier transform; σ may usually be taken zero in the generalized Fourier sine and cosine transforms, thus converting them to ordinary Fourier transforms; and $\mathfrak{L}_{(1/\omega)}$, $\mathfrak{F}_{s(1/\omega)}$, etc., denote that the independent variable of the transform is ω^{-1}. It may be noted that these equations may be formally inverted by multiplying through with the proper inverse transform operators, yielding $G(\tau)$ in terms of transforms of $\omega Q(i\omega)$, $\omega J(\omega)$, and $\omega H(\omega)$. The results obtained are difficult to apply, however, and may not hold in general.

Next, we shall consider how many of our earlier integral transform equations may be inverted by means of Mellin transform relations. We shall denote the Mellin transforms of pertinent variables by lower case letters. If $\mathfrak{M}$ is the Mellin transform integral operator,[47] then $\mathfrak{M}[J(\omega)]=j(s)$, where s is the complex independent variable of the Mellin transform. Our results depend upon an equation given by Titchmarsh.[74] If a transform relation can be written in the form

$$W(x)=\int_0^\infty K(xy)F(y)dy, \quad (76)$$

it is then easy to show that operating on both sides of this equation with $\mathfrak{M}$ yields formally

$$w(s)=k(s)f(1-s), \tag{77}$$

where $k(s)$ is the Mellin transform of the kernel $K(xy)$ written as $K(x)$. This result depends upon an interchange of the order of integration after applying the Mellin transform.

We may now take the Mellin transform of all of our pertinent previous integral relations, then use (77) to obtain the corresponding Mellin-transform relation. This procedure applied to our previous results leads to a large number of equations, most of which are summarized in their most useful forms in Appendix V. These equations show that if a function such as $g(s)$ is known, it is possible to obtain a desired quantity such as $j(s)$, $a(s)$, $q_p(s)$ etc., by algebraic manipulation alone. Then, if the inverse Mellin transform of the desired quantity can be obtained, either through the use of tables[2] or by complex integration,[47] the original integral equation relating the known and the unknown quantities, e.g., $G(\tau)$ and $Q(p)$, will have been inverted. This procedure is often very useful since it makes available another table of transforms as an aid to inversion; it fails, of course, when the inverse transform does not exist or cannot be found.

The relations between the Mellin transform quantities $h(s)$ and $j(s)$ and $q_p(s)$ are of particular interest. It will be noted that these quantities are related as the sides of a right triangle with the included angle between $q_p(s)$ and $j(s)$ being $\pi s/2$. Similarly, $J(\omega)$ and $H(\omega)$ are at right angles in the complex $Q(i\omega)$ plane. Here, of course, the angle included between $|Q(i\omega)|$ and $J(\omega)$ depends on ω.

Next, we may make use of a well-known Mellin-transform convolution integral[75] which states that

$$\mathfrak{M}^{-1}[f(s)k(s)]=\int_0^\infty F\left(\frac{x}{y}\right)K(y)\frac{dy}{y}. \tag{78}$$

This relation may be used to derive some new equations from the relations given in Appendix V. It may first be mentioned that applying (78) to $a(s)=\Gamma(s)g(s)$ yields (63) immediately. In a similar fashion, (66) and (72) may be proved. All the previously given integral transforms, such as the Kronig-Kramers relations, may be regained in the same way.

To apply (78) in order to derive further relations, we shall first use the following inverse Mellin transforms

$$\mathfrak{M}^{-1}\left[\sin\frac{\pi s}{2}\right]=\tfrac{1}{2}[\Delta(x-i)+\Delta(x+i)], \tag{79}$$

$$\mathfrak{M}^{-1}\left[\cos\frac{\pi s}{2}\right]=\frac{i}{2}[\Delta(x-i)-\Delta(x+i)]. \tag{80}$$

In the above equations, we attach only a formal meaning to quantities like $\Delta(x-i)$. This operator, which may be termed a complex delta function, is analogous to an ordinary delta function except that the real path of integration does not include the point in the complex plane (off the real axis) where its argument is zero. For example, we so define this quantity that

$$\mathfrak{M}[\Delta(x-i)]=\int_0^\infty x^{s-1}\Delta(x-i)dx \equiv[e^{i\pi/2}]^{s-1}=-ie^{i\pi s/2}. \tag{81}$$

Now, on using the above results, Eq. (78), and Eq. (11) of Appendix V, we obtain the new relations

$$G(\tau)=\frac{i}{\pi\tau}[J(i\tau^{-1})-J(-i\tau^{-1})]$$

$$G(\lambda^{-1})=\frac{2\lambda}{\pi}\,\mathrm{Im}[J(\lambda e^{-i\pi/2})] =\lim_{\epsilon\to 0}\frac{2\lambda}{\pi}\,\mathrm{Im}[J(-i\lambda+\epsilon)], \tag{82}$$

$$G(\tau)=\frac{1}{\pi\tau}[H(i\tau^{-1})+H(-i\tau^{-1})]$$

$$G(\lambda^{-1})=\frac{2\lambda}{\pi}\,\mathrm{Re}[H(\lambda e^{-i\pi/2})] =\lim_{\epsilon\to 0}\frac{2\lambda}{\pi}\,\mathrm{Re}[H(-i\lambda+\epsilon)]. \tag{83}$$

It will be noted that these equations are similar to (66) and (72). Further, their validity again depends upon the possibility of analytic continuation of $J(\omega)$ and $H(\omega)$ into the complex plane. The above complex delta function method of obtaining Eqs. (82) and (83) is only one of several possible methods. We give it here because we shall make further use of such functions later. Equation (83), expressed in a different form, was first obtained using other methods by Fuoss and Kirkwood.[69] Both equations have been given previously and discussed by Gross.[10,11,13]

In Fig. 2, we show as a flow diagram some of the relations which spring from the original connection be-

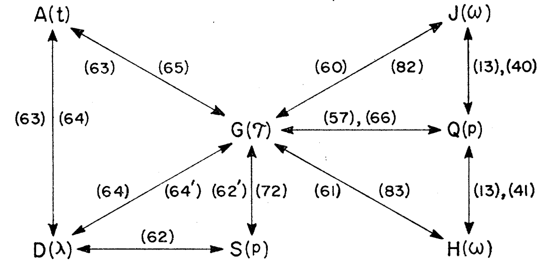

FIG. 2. Diagram showing the connections between the relaxation-time distribution function $G(\tau)$ and other quantities characterizing the linear system having the $G(\tau)$ distribution.

tween $G(\tau)$ and $Q(p)$. As in Fig. 1, the numbers on this diagram are the equation numbers of the connecting relations. Since we have assumed that constants such as $J(\infty)$ and $P(\infty)$ have been removed by normalization, the use of dotted lines as in Fig. 1 is unnecessary here.

Earlier, we mentioned that the hypothesis of a distribution of relaxation times has been extensively used in interpreting dielectric constant measurements. For example, Cole and Cole[76] suggested the use of the network function

$$Q(i\omega)=[1+(i\omega\tau_0)^{1-\alpha}]^{-1} \tag{84}$$

for dielectric systems with a distribution of a relaxation times, and the real and imaginary parts of this function have been found very useful in interpreting experimental results. There is a single relaxation time when the constant α is zero and an infinitely broad distribution when α is unity. It is obvious that the $Q(p)$ function corresponding to the above form of $Q(i\omega)$ is not an analytic function of the complex variable p when $0<\alpha<1$. As a consequence, the $J(\omega)$ and $H(\omega)$ functions derived from (84) represent a noncausal system. The situation may be saved by putting in the necessary $|\omega|$ and sign ω factors in $J(\omega)$ and $H(\omega)$ to force them to have the proper parity. They will then not be the real and imaginary parts of the nonanalytic $Q(p)$. The actual dependence of $J(\omega)$ and $H(\omega)$ on ω for real positive ω will not be changed by these additions, and they may thus still be used for interpreting dielectric constant measurements. Further, forcing $J(\omega)$ and $H(\omega)$ to have the correct parity in ω ensures that the integral transforms yielding these functions from $G(\tau)$ will give the correct results and that the $G(\tau)$ obtained from $J(\omega)$ or $H(\omega)$ will specify a causal system, as it should for physical realizability.

Later, Davidson and Cole[77] presented the function

$$Q(i\omega)=[1+i\omega\tau_0]^{-\beta}, \tag{85}$$

and showed that it too was useful in analysis of dielectric relaxation measurements. In this equation, the constant β may range from zero to unity. It is noteworthy that although this function cannot explain all data showing a distribution of relaxation times, it leads to $J(\omega)$ and $H(\omega)$ functions of proper parity, its $Q(p)$ is analytic, and it therefore can represent a class of physically realizable systems.

Fuoss and Kirkwood[69] suggested for $H(\omega)$ the function

$$H(\omega)=2[(\omega/\omega_m)^{\alpha}+(\omega_m/\omega)^{\alpha}]^{-1}, \tag{86}$$

where ω_m and α are constants and $0\leq\alpha\leq1$. Here, $H(\omega)$ is obviously not an odd function of ω inside this range. Macdonald[16] has used the above $H(\omega)$ to obtain $J(\omega)$ by means of one of the Kronig-Kramers transforms for several fractional values of α. The resulting $J(\omega)$ functions are neither even nor odd as written. Nevertheless, the fact that ω enters one of the Kronig-Kramers relations only as ω^2 and the other only as $\omega f(\omega^2)$ shows that they must, of necessity, yield only even or odd functions of ω, as the case may be. This result is independent of the form of $J(\omega)$ or $H(\omega)$ since these functions appear under the integral sign only as functions of the independent variable y. Hence, Macdonald's results should have been written to show that $J(\omega)$ was in fact even in ω. Again, such addition will not change the functional dependence of $J(\omega)$ on ω for real, positive ω. These results show that for the $H(\omega)$ of (86) to represent a physical system, it must be rewritten in a form which makes it of odd parity. In such a form, it and the $J(\omega)$ results with corrected parity will satisfy the Kronig-Kramers relations.

As we have noted earlier, the Kronig-Kramers relations are usually derived on the assumption that for the function considered, $S(p)$ or $Q(p)$, there are no poles whatsoever in the right half of the complex p-plane. Elsewhere,[31] the present authors have derived these relations by means of Mellin transforms directly from the integral relations between $J(\omega)$ and $H(\omega)$ and $G(\tau)$. This derivation makes no explicit use of the condition of analyticity, and it is therefore pertinent to inquire to what extent the Kronig-Kramers relations hold for nonanalytic functions. In a passive system, $S(p)$ and $Q(p)$ must be analytic in the finite, nonzero part of the right half plane, but they need not be if the system is active.

We may begin by distinguishing for our present purposes three types of nonanalyticity. A function may not be analytic only at $p=0$ or $p=\infty$, it may not be analytic at isolated points in the right half plane (excluding $p=0$ and $p=\infty$), and it may not be analytic anywhere in this half of the p plane. The first type is represented by the functions $S(p)=p^{-1}$ and p, corresponding to an ideal (passive) capacitance and inductance, respectively. Now, it is easy to show the real and imaginary parts of the $S(i\omega)$ functions corresponding to these choices satisfy the Kronig-Kramers relations (and our other integral transform relations) as long as the pertinent integrals converge. For $S(p)=p^{-1}$, $P(\omega)=\pi\delta(\omega)$, and $T(\omega)=-\omega^{-1}$. Equation (29′) yields, if one uses this value of $T(\omega)$,

$$P(\omega)=\frac{2}{\pi}\int_0^{\infty}\frac{dy}{y^2-\omega^2}=\pi\delta(\omega). \tag{87}$$

Similarly, if we use $P(\omega)$ in (30′), we find (on changing the limits of integration to accommodate the delta function)

$$T(\omega)=-\omega\int_{-\infty}^{\infty}\frac{\delta(y)dy}{\omega^2-y^2}=-\omega^{-1}. \tag{88}$$

For $S(p)=p$, on the other hand, we have $P(\omega)=0$, $T(\omega)=\omega$, and $J(\infty)=1$. We find that (29′) is not convergent but that both (30′) and (35) hold.

The second type of nonanalyticity is exemplified by the function $S(p)=(p-1)^{-1}$ which is not analytic at $p=1$. A function of this type is found to satisfy the Kronig-Kramers relations only when a mathematical artifice is employed. If we write its $S(i\omega)$ as $-[(-1)^2+\omega^2]^{-1}-i\omega[(-1)^2+\omega^2]^{-1}$, we see that it involves a negative resistance. The impulse response is an ever-increasing function. Hence, the system is unstable. On applying the Kronig-Kramers relations to $P(\omega)$ and $T(\omega)$, we obtain, for example, the integral

$$T(\omega)=\frac{2\omega}{\pi}\int_0^\infty \frac{dy}{[(-1)^2+y^2][\omega^2-y^2]}=\frac{-\omega}{(-1)^2+\omega^2}. \quad (89)$$

The correct result is only obtained if we carry $(-1)^2$ in this form so that $[(-1)^2]^{\frac{1}{2}}$ becomes -1. Under these conditions, the other Kronig-Kramers pair yields the correct result for $P(\omega)$ as well.

It must not be supposed that all systems involving negative elements require the above artifice. A system may involve a negative resistance and still be stable. An example is $S(p)=(p-a)/(p+b)$, whose impulse response is $B(t)=\delta(t)-(a+b)e^{-bt}$. This $S(p)$ has no poles in the right half of the p plane but its $P(\omega)$ is negative for $\omega<(ab)^{\frac{1}{2}}$. All the integral transform relations hold for this function. Finally, it is easy to establish that the Kronig-Kramers relations cannot hold for a function which is nonanalytic in the entire plane or entire right half plane. Such a function is $S(p)=(p^*)^{-1}=[\sigma-i\omega]^{-1}$.

The prior discussion of relaxation-time distributions has been in terms of distributions of series RC-type time constants as indicated by (57). For a single time constant, we have used the $Q(p)$ related to the admittance of a capacitor and resistor in series and have obtained a distribution of relaxation times by weighting this function with the distribution function $G(\tau)$, which will now be designated

$$G_{\binom{R}{C}}(\tau).$$

It is desirable to point out that the above arrangement is by no means the only possibility. When we are concerned with a distribution over values of a single constant, there are, in fact, three other possibilities.

We may write an equation identical to (57) but with the $Q(p)$ for a single relaxation time equal to $Z(p)/p$ instead of $Y(p)/p$ and interpret $Z(p)$ as the system function of a resistance and inductance in parallel. The distribution function for this case may be designated as $G_{(RL)}(\tau)$. All the succeeding equations will be the same but analysis will then be in terms of parallel RL circuits having different L/R time constants.

Another possibility is to analyze a system into branches containing a capacitor and resistor in parallel. Instead of (57) we must then write

$$S(p)=Z(p)=\int_0^\infty \frac{\hat{G}_{(RC)}(\tau)d\tau}{1+p\tau}. \quad (90)$$

When only a single relaxation time $\tau_0=RC$ is present, $\hat{G}_{(RC)}(\tau)=\delta(\tau-\tau_0)$ and $Z(p)$ is the input impedance of the capacitance and resistance in parallel. The caret over $\hat{G}_{(RC)}$ is intended to indicate that it is directly related to a system function rather than a network function $Q(p)$. The fourth possibility is to analyze in terms of series resistance-inductance branches. Then the pertinent equation becomes

$$S(p)=Y(p)=\int_0^\infty \frac{\hat{G}_{\binom{R}{L}}(\tau)d\tau}{1+p\tau}. \quad (91)$$

The discussion thus far has dealt only with systems containing energy storage of a single type, inductive or capacitative or their mechanical analogues. When both types are present in the same system, as in the RLC circuit, there may be a distribution of damping time constants τ and an entirely separate distribution of resonant frequencies ω_0 as well. The distribution function must then be a function of both τ and ω_0 and must be connected to a system or network function by means of a double integral over these variables. Elsewhere,[32] we have shown that if the distribution function is such that the double integral converges, the real and imaginary parts of the network function with which it is connected satisfy the Kronig-Kramers relations. Further, we shall show later that a simple RLC circuit without a distribution of τ or ω_0 may be formally represented by means of a series RC distribution function by using the artifice of the complex delta function.†

The above discussion of distribution functions has been tacitly restricted to input impedances or admittances and their corresponding network functions. Sometimes, however, it is desirable to obtain the distribution of relaxation times connected with a transfer ratio such as amplifier gain, filter response, etc., instead of that of the previously considered two-terminal functions such as complex dielectric constants or magnetic susceptibilities. Here, therefore, we shall briefly consider the two simplest connections between a transfer ratio and its corresponding distribution function, $G_T(\tau)$.

The cases to be discussed are those in which the transfer ratio is composed of the additive responses of simple RC filters of either the low- or high-pass type. Since the transfer ratio of such a low-pass filter may be written in the form $S(p)=(1+p\tau_0)^{-1}$ where $\tau_0=RC$, the transfer ratio of the corresponding low-pass device with a continuous or discontinuous distribution of relaxation times is simply given by our basic equation (57) with $Q(p)$ replaced by $S(p)$ and $G(\tau)$ by $G_{TL}(\tau)$, where the subscript L is added to denote a distribution function connected with low-pass response. It is thus evident that all the previous distribution function equations

† Note added in proof.—B. Gross, Lineare Systeme, Supplement to Nuovo cimento 3, 235 (1956), has recently treated RLC systems in a different, more general manner.

apply to the present case with only minor changes of notation necessary.

The high-pass case is slightly more complicated since its elemental transfer ratio is $(p\tau_0)/(1+p\tau_0)$, where τ_0 again equals RC. On averaging such a function over a distribution of relaxation times, $G_{TH}(\tau)$, we obtain

$$S(p)=p\int_0^\infty \frac{\tau G_{TH}(\tau)d\tau}{1+p\tau},$$

or

$$Q(p)=\int_0^\infty \frac{\tau G_{TH}(\tau)d\tau}{1+p\tau}=\int_0^\infty \frac{\mathcal{G}_{TH}(\tau)d\tau}{1+p\tau}, \quad (57''')$$

where $\mathcal{G}_{TH}(\tau)=\tau G_{TH}(\tau)$. In both the high- and low-pass cases we have assumed, as usual, that the $G_T(\tau)$ functions are normalized to unity. Since we have once more ended with a form of the basic equation (57), the earlier equations of this section again apply.

It should be noted that the present results refer specifically to transfer ratios which show limiting frequency response slopes no greater than the 6 db/octave obtainable from a single RC section. When the frequency response curve shows slopes less than or equal to this value, the response may be correlated with a distribution of RC relaxation times of the present type. On the other hand, when greater limiting slopes are present, the relaxation-time distribution function obtained by straightforward application of the above formulas may be negative over part of its range. When such behavior occurs, it is clear that a more complicated initial transfer ratio than that of a single RC section should be used in establishing a connection between the transfer ratio and its distribution function.

IV. ILLUSTRATIVE EXAMPLES

Throughout the previous sections of this work we have included simple examples where they seemed pertinent. In this concluding section, we shall present a few more examples to illustrate the application and applicability of the preceding formulas.

The first example is that of a series RC circuit, for which the normalized network function (derived from an admittance) is $Q(p)=[1+\tau_0 p]^{-1}$, where $\tau_0=RC$. The indicial admittance is readily found from (11) to be $A(t)=\tau_0^{-1}\exp[-t/\tau_0]$. From (65) this result leads to $G(\tau)=\tau^{-1}\tau_0^{-1}\delta(\tau^{-1}-\tau_0^{-1})=\delta(\tau-\tau_0)$ using Appendix II. As expected, the spectrum is a single line with the time constant τ_0. If, alternatively, we make use of (66′), we find $D(\lambda)=\lim \epsilon\to 0\ \{\epsilon\tau_0^{-1}\pi^{-1}/[\epsilon^2+\tau_0^{-2}(1-\lambda\tau_0)^2]\}=\tau_0^{-1}\delta(\lambda-\tau_0^{-1})$. The corresponding $G(\tau)$ is that above. A slightly different result is obtained if we calculate $G(\tau)$ from $J(\omega)$ or $H(\omega)$ using (82) or (83). The former yields $G(\tau)=\delta(\tau-\tau_0)+\delta(\tau+\tau_0)$. When $G(\tau)$ is calculated from $H(\omega)$, the second term again appears but is then negative. It is clear that this term will contribute nothing to the integrals involving $G(\tau)$ because integration extends only over the range of positive τ. The second term may be eliminated by arbitrarily defining $G(\tau)=0$ for $\tau<0$. It is interesting to note that in this example the complete $G(\tau)$ derived from the even $J(\omega)$ is even in τ while that derived from $H(\omega)$ is odd in τ.

Next, we shall consider the input impedance $S(p)=\tanh(a+bp)^{\frac{1}{2}}/(a+bp)^{\frac{1}{2}}$. For convenience, we shall not normalize this quantity to unity at $p=0$. This $S(p)$ is the short-circuit input impedance of an idealized transistor;[78] thus, it represents a distributed system. Since $S(p)$ is a meromorphic function, it may be written first as an infinite product involving its poles in the complex p plane, then as an infinite series of partial fractions.[79] This transformation leads to

$$S(p)=(2/b)\sum_{n=0}^{\infty}[p+(a/b)+(\pi^2/b)(n+\tfrac{1}{2})^2]^{-1}. \quad (92)$$

On using (10) to obtain $B(t)$, we find

$$B(t)=(2/b)\sum_{n=0}^{\infty}\exp\{-[a+\pi^2(n+\tfrac{1}{2})^2](t/b)\}. \quad (93)$$

This result, which may also be written as a Jacobian theta function, suggests that it would be pertinent to analyze the system into parallel RC branches. Since we are here dealing with an impedance, we may use Eq. (90) to obtain the distribution function $\hat{G}_{(RC)}(\tau)$. We shall work first with the corresponding $\hat{D}(\lambda)$, which may most readily be found from the relation $\hat{D}(\lambda)=\mathcal{L}^{-1}[B(t)]$ corresponding to (64). We obtain

$$\hat{D}(\lambda)=(2/b)\sum_{n=0}^{\infty}\delta\{\lambda-[(a/b)+(\pi^2/b)(n+\tfrac{1}{2})^2]\}. \quad (94)$$

The corresponding $\hat{G}(\tau)$ is

$$\hat{G}(\tau)=(2/b\tau)\sum_{n=0}^{\infty}\left[\frac{a}{b}+\frac{\pi^2}{b}(n+\tfrac{1}{2})^2\right]^{-2}\times\delta\left\{\tau-\left[\frac{a}{b}+\frac{\pi^2}{b}(n+\tfrac{1}{2})^2\right]^{-1}\right\}$$

$$=\left(\frac{2}{b}\right)\sum_{n=0}^{\infty}\left[\frac{a}{b}+\frac{\pi^2}{b}(n+\tfrac{1}{2})^2\right]^{-1}\times\delta\left\{\tau-\left[\frac{a}{b}+\frac{\pi^2}{b}(n+\tfrac{1}{2})^2\right]^{-1}\right\}. \quad (95)$$

This result shows that even though we are dealing with a distributed system, its input impedance may be represented by an infinite number of parallel RC's. Such infinite line spectra are characteristic of meromorphic functions, which are generalizations of algebraic rational functions. The latter involve only a finite number of relaxation times. In the above example, the number of relaxation times at $\tau=0$ is zero; as τ increases, the number (or weight of the line) increases until it reaches a maximum at $\tau=b/[a+(\pi/2)^2]$. There are no relaxation times greater than this value. Note that $\hat{G}(\tau)$ is entirely positive; it is not normalized to unity because $S(0)$ was not so normalized.

Since it is not in all cases practical to carry out the inverse Laplace transformation which led above to $\hat{D}(\lambda)$, it is sometimes necessary to use Eq. (66′) or its equivalent instead. Here, we may use (66′) with the function Q replaced by S. We shall show how such an analysis is carried through for the present input impedance as an illustration of the method for more complicated cases than considered previously. We must evaluate

$$\hat{D}(\lambda)=\lim_{\epsilon\to 0}\frac{1}{\pi}\,\mathrm{Im}\left[\frac{\tanh(a-b\lambda-ib\epsilon)^{\frac{1}{2}}}{[a-b\lambda-ib\epsilon]^{\frac{1}{2}}}\right]. \tag{96}$$

Since ϵ will vanish, we may expand the square root and obtain $[a-b\lambda-ib\epsilon]\to(\alpha-i\beta)$, where $\alpha=[a-b\lambda]^{\frac{1}{2}}$, $\beta=b\epsilon/2\alpha$. Since the quantity β will also disappear as ϵ goes to zero, it may be treated as an infinitesimal and only first-order terms in β retained. The process of rationalizing (96) and taking the limit must be carried out very carefully; for example, it is incorrect to rationalize the tanh function separately.[80] Instead, we must first rationalize the denominator, written in the form $(\alpha-i\beta)\cosh(\alpha-i\beta)$. We find that the imaginary part of (96) yields, for sufficiently small β,

$$\hat{D}(\lambda)=\lim_{\beta\to 0}\frac{\sinh\alpha\cosh\alpha-\alpha}{\pi[\alpha\sinh\alpha+\cosh\alpha]^2}\cdot\frac{\beta}{\left[\beta^2+\left(\dfrac{\alpha\cosh\alpha}{\alpha\sinh\alpha+\cosh\alpha}\right)^2\right]}$$

$$=\frac{\sinh\alpha\cosh\alpha-\alpha}{[\alpha\sinh\alpha+\cosh\alpha]^2}\delta\left[\frac{\alpha\cosh\alpha}{\alpha\sinh\alpha+\cosh\alpha}\right]. \tag{96′}$$

The delta function may be further simplified using the results of Gross and Pelzer[81] given in Appendix II. The zeros of the delta function occur when α and $\cosh\alpha$ are zero. However. the factor multiplying the delta function is also zero for $\alpha=0$; hence this value of α does not contribute to $\hat{D}(\lambda)$. We may therefore treat the quantity $\alpha/[\alpha\sinh\alpha+\cosh\alpha]$ inside the delta function as a constant and remove it to the outside, obtaining

$$\hat{D}(\lambda)=\frac{\sinh\alpha\cosh\alpha-\alpha}{\alpha[\alpha\sinh\alpha+\cosh\alpha]}\delta[\cosh\alpha]$$

$$=\frac{-\delta(\cosh\alpha)}{\alpha\sinh\alpha}=\frac{-\delta[\cosh(a-b\lambda)^{\frac{1}{2}}]}{[a-b\lambda]^{\frac{1}{2}}\sinh(a-b\lambda)^{\frac{1}{2}}}. \tag{96″}$$

The second equation follows from the fact that an integral over $\hat{D}(\lambda)$ can only be different from zero when $\cosh\alpha=0$. Now, on applying the Gross-Pelzer expansion formula to (96″), we find

$$\hat{D}(\lambda)=\sum_{n=0}^{\infty}\frac{(2/b)(a-b\lambda_n)^{\frac{1}{2}}\delta(\lambda-\lambda_n)}{(a-b\lambda_n)^{\frac{1}{2}}\sinh^2(a-b\lambda_n)^{\frac{1}{2}}}$$

$$=\frac{2}{b}\sum_{n=0}^{\infty}\delta(\lambda-\lambda_n), \tag{96‴}$$

where $\lambda_n=(a/b)+(n+\frac{1}{2})^2/b$. Comparison of the final result with that of (94) shows that they are identical.

The next example is concerned with what may be termed the complete Lorentz dispersion formula. Consider a system whose un-normalized $Q(i\omega)$ function is a complex dielectric constant $\epsilon=\epsilon_1-i\epsilon_2$. Then the corresponding Lorentz dispersion $J(\omega)$ function is[82,83]

$$J(\omega)\equiv\epsilon_1=\epsilon_\infty+\tfrac{1}{2}(\epsilon_0-\epsilon_\infty)\times\left[\frac{1+\omega_0(\omega+\omega_0)\tau_0^2}{1+(\omega+\omega_0)^2\tau_0^2}+\frac{1-\omega_0(\omega-\omega_0)\tau_0^2}{1+(\omega-\omega_0)^2\tau_0^2}\right]. \tag{97}$$

For convenience $J(\omega)$ has been written in terms of conventional dielectric quantities; however, the same type of dispersion may occur in magnetic systems as well.[84–86] As long as ω_0 is not zero, it represents resonant absorption, as distinguished from the simple nonresonant absorption of the Debye type to which it reduces when $\omega_0=0$.

In order to obtain the indicial admittance $A(t)$, Eq. (27) may be applied and tabulated transforms used.[2] The result is

$$A(t)=\epsilon_\infty\delta(t)+(\epsilon_0-\epsilon_\infty)\tau_0^{-1}e^{-t/\tau_0}\times[\cos\omega_0 t+\omega_0\tau_0\sin\omega_0 t], \tag{98}$$

showing that the system response to a unit step function is a damped sinusoid for $\omega_0>0$, apart from the delta function response to the ϵ_∞ term. Now (98) may be used in (24) to give $H(\omega)$. We find

$$H(\omega)\equiv\epsilon_2=\tfrac{1}{2}(\epsilon_0-\epsilon_\infty)\times\left[\frac{\omega\tau_0}{1+(\omega+\omega_0)^2\tau_0^2}+\frac{\omega\tau_0}{1+(\omega-\omega_0)^2\tau_0^2}\right]. \tag{99}$$

The quantity $\omega H(\omega)\equiv P(\omega)$ is proportional to the power loss of the system. The maximum value of $H(\omega)$ occurs at $\omega=\omega_m=\tau_0^{-1}[1+(\omega_0\tau_0)^2]^{\frac{1}{2}}$. Next, we may use (98) to obtain $Q(p)$ and $S(p)$. Equation (9) yields

$$Q(p)=\epsilon_\infty+(\epsilon_0-\epsilon_\infty)\left[\frac{\tau_0^{-1}p+(\omega_0^2+\tau_0^{-2})}{p^2+2\tau_0^{-1}p+(\omega_0^2+\tau_0^{-2})}\right]. \tag{100}$$

Note that when $\omega_0=0$, $Q(p)$ may be factored to yield the Debye dispersion formula. The network function $Q(p)$ here represents the dielectric constant of the system as a function of the complex frequency variable p.

The system function $S(i\omega)$ corresponding to the given $Q(i\omega)$ is the input admittance of the system or a quantity proportional thereto, as was tacitly assumed by the designation of $A(t)$ as the indicial admittance.

The system function will be a physically realizable positive-real algebraic function of p. It may, therefore, be represented by a finite number of lumped resistances, capacitances, and inductances. In view of the great generality and physical importance of the complete Lorentz dispersion formulas, it is pertinent to ask what the structure and element values would be of a lumped constant system which had the same system function, indicial admittance, etc. To obtain such a representation we may write the system function as follows

$$S(p)=Y(p)=p\epsilon_\infty+\tfrac{1}{2}(\epsilon_0-\epsilon_\infty)\left[\frac{2\tau_0^{-1}p^2+(\omega_0^2+\tau_0^{-2})p}{p^2+2\tau_0^{-1}p+(\omega_0^2+\tau_0^{-2})}+\frac{(\omega_0^2+\tau_0^{-2})p}{p^2+2\tau_0^{-1}p+(\omega_0^2+\tau_0^{-2})}\right]. \quad (101)$$

In this form, the last two terms on the right may be readily identified as parallel branches each involving resistance, capacitance, and inductance. It will be noted that the input admittance is written in terms of four (measurable) constants ϵ_0, ϵ_∞, τ_0, and ω_0 which characterize the system. Further, the numerical factor which converts dielectric constants into capacitances has been taken to be unity for convenience in writing (101). If we continue to represent capacitance in terms of its corresponding dielectric constant, the element values and structure of Fig. 3 are readily derived from $S(p)$ written in the form (101). This is not the only possible lumped constant representation of complete Lorentz dispersion, but it is probably one of the simpler and more useful forms.

When we exclude ϵ_∞, the two capacitances are equal and so are the two inductances. Because of such equality, there is only one resonant frequency. The limiting low-frequency capacitance is ϵ_0; the high frequency limiting value is ϵ_∞. It will be noted that in the limit $\omega_0\rightarrow 0$, the circuit of Fig. 3 does not reduce to the simple Debye form of a resistance $\tau_0/[\epsilon_0-\epsilon_\infty]$ in series with a capacitance $[\epsilon_0-\epsilon_\infty]$; instead all three branches remain, and the inductances are not zero. For such nonresonant absorption, however, each branch is critically damped and the over-all circuit has the same system function as that of the above resistance and capacitance in series. In this limit, it is not possible to determine from measurements of ϵ_0, ϵ_∞, and τ_0 whether the system contains inductance (or mass in the vibrating spring case) and is critically damped, or whether it consists only of a resistance and capacitance, or their mechanical equivalents. The representations are electrically equivalent and either is valid.

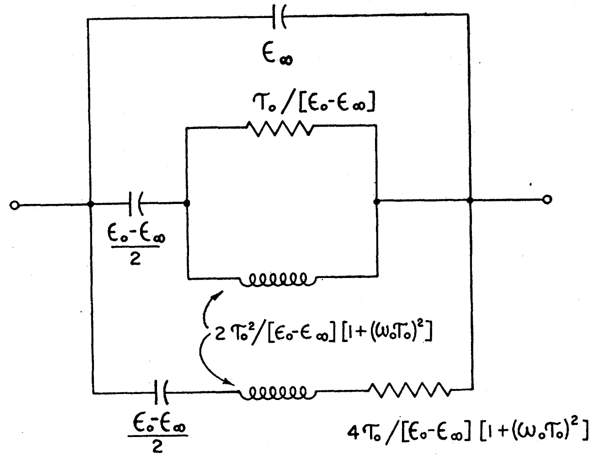

FIG. 3. A lumped-element circuit realization of a system exhibiting complete Lorentz dispersion.

The complete Lorentz dispersion formulas admit either under-damping or critical damping; over-damping is not allowed by the equations. On the other hand, what may be termed simple Lorentz dispersion[87] allows all three possibilities. Such dispersion is obtained, for example, from a system such as an electron of mass m bound to an equilibrium position by Hooke's law forces and subject to a damping force proportional to velocity. It may be represented electrically by a resistance, capacitance, and inductance all in series. We have shown elsewhere how the Kronig-Kramers equations may be applied to such a system.[32]

Our present expressions for resonant absorption apply no matter what the ratio of absorption line width to resonant frequency. They may be usefully simplified in the often encountered case for which the ratio is very small by essentially neglecting terms involving $(\omega+\omega_0)$ compared to those involving $(\omega-\omega_0)$.[18] The following forms for $H(\omega)$ have been used,

$$H(\omega)=A\left[\frac{\omega\tau_0}{1+(\omega-\omega_0)^2\tau_0^2}\right]\simeq A\left[\frac{\omega_0\tau_0}{1+(\omega-\omega_0)^2\tau_0^2}\right], \quad (102)$$

where A is a constant. It will be noted that even the first form given is not correctly of odd parity in ω. The proper parity of this equation may be obtained, however, by writing ω as $|\omega|$ in the denominator. Although the above result was derived only for the case of absorption lines narrow compared to the resonant frequency, it has been applied by Lacroix to the case where the line width is comparable to the resonant frequency.[19] Lacroix applied the Kronig-Kramers transform equation (31′) to the first form of (102) and obtained the corresponding $J(\omega)$. The result was properly even in ω but was quite complicated and involved logarithmic functions of ω and ω_0. We wish to point out that the use of (102) is not theoretically justified in the wide line case and that it would have been far preferable to use the complete Lorentz dispersion formulas instead. The appearance of logarithmic terms implies that there does not exist a simple electric circuit of lumped elements equivalent to Lacroix's system, as there is for the complete Lorentz dispersion system. Further, the $J(\omega)$ of the latter, given by (97), is far simpler than that obtained by Lacroix.

Finally, we wish to examine the consequences of a brute-force application of the distribution of relaxation time formulas (57) through (65) to the complete Lorentz dispersion equations, which as we have seen, involve inductances, or their mechanical equivalents, as well as capacitances. To obtain

$$G_{\binom{R}{C}}(\tau)=G(\tau),$$

it proves simplest to substitute (98) in (65). We obtain

$$\begin{aligned}G(\tau)=&\frac{(\epsilon_0-\epsilon_\infty)}{4\pi\tau\tau_0}\int_0^\infty\Bigg\{(1+i\omega_0\tau_0)\\&\times\exp\left[i\omega\left(\frac{1}{\tau}-\frac{1}{\tau_0}-i\omega_0\right)\right]\\&+(1-i\omega_0\tau_0)\exp\left[i\omega\left(\frac{1}{\tau}-\frac{1}{\tau_0}+i\omega_0\right)\right]\Bigg\}d\omega\\=&\frac{(\epsilon_0-\epsilon_\infty)}{2\tau\tau_0}\Bigg[(1+i\omega_0\tau_0)\Delta\left[\frac{1}{\tau}-\frac{1}{\tau_0}-i\omega_0\right]\\&+(1-i\omega_0\tau_0)\Delta\left[\frac{1}{\tau}-\frac{1}{\tau_0}+i\omega_0\right]\Bigg]\\=&\frac{(\epsilon_0-\epsilon_\infty)\tau_0}{2\tau}\left\{\frac{\Delta\left[\tau-\dfrac{\tau_0}{1+i\omega_0\tau_0}\right]}{(1+i\omega_0\tau_0)}+\frac{\Delta\left[\tau-\dfrac{\tau_0}{1-i\omega_0\tau_0}\right]}{1-i\omega_0\tau_0}\right\}\\=&(\epsilon_0-\epsilon_\infty)\,\mathrm{Re}\left\{\Delta\left[\tau-\frac{\tau_0}{1+i\omega_0\tau_0}\right]\right\}.\end{aligned}\tag{103}$$

In obtaining this result, we have used the results of Appendix II but have written the resulting delta functions as $\Delta(x)$ instead of $\delta(x)$ because of the appearance of a complex argument. When (103) is substituted in (57), we immediately obtain the result (100) on remembering that the $\Delta(x)$ operator is the same as an ordinary delta function except that it involves a complex argument yet a real path of integration. It is thus only a formal substitution operator.‡

‡ Note added in proof.—H. Pelzer, Technical Report L/T332, British Electrical Research Association, 1955, has recently given a rigorous definition of the delta operation in the complex plane. The resulting complex delta function differs from the present Δ function.

To conclude this section, we have presented in Table I some of the examples already discussed in the text as well as some new results. All of the functions appearing in this table are defined in reference 2. For example, $K_0(x)$ is Macdonald's modified Bessel function. Where possible, we have normalized so that $J(0)=1$ and $J(\infty)=0$. The constant τ_0 must, in general, be greater than zero. Further, it is usually necessary that $t\geqslant 0$ and $\mathrm{Re}(p)>0$ or $-\tau_0$ for all the transforms of a given row to hold. For those positions filled with a line, we have reasons to believe the indicated quantity is zero, inapplicable, or infinite. These reasons are associated either with the presence of unbounded integrals or the impossibility of carrying out the inverse Laplace (or Mellin) transforms for certain classes of functions.[4,88] Thus, for certain functions such as $A(t)=\sin(t/\tau_0)/t$, the corresponding $D(\lambda)=\mathfrak{L}^{-1}[A(t)]$ is zero (except for a possible set of measure zero) or nonexistent, and the analysis of the system in terms of a distribution of relaxation times is impossible except perhaps formally in terms of the artifice of the complex delta function. The systems for which this is the case are not relaxation systems.

The empty positions in the table represent situations where we have not obtained explicit results for the corresponding quantities but have no especial reason to believe them nonexistent. It will be noted that in several cases we have given values for $g(s)$ even when there is no corresponding $G(\tau)$. These values of $g(s)$ were obtained by means of the Mellin transform relation $g(s)=[\Gamma(s)]^{-1}a(s)$, $(\mathrm{Re}s>0)$. The existence of a $g(s)$ obtained in this fashion does not imply the existence of $G(\tau)$; it is, however, what the Mellin transform of $G(\tau)$ would be if $G(\tau)$ existed.

Since $J(0)=\int_0^\infty G(\tau)d\tau=g(1)$, it is apparent that $g(1)=1$ is the normalization condition. All values of $g(s)$ in the table reduce to unity for $s=1$ except that of No. 18 for which $J(\omega)$ and $G(\tau)$ cannot be normalized. Note that for Eq. (28) to hold for No. 19, it is necessary to adopt the convention that $\lim_{\omega\to\infty}$ of $J(\omega)=\sin(|\omega|\tau_0)$ is zero. It will be readily apparent that the results of the table apply as well after the following change of nomenclature: $Q(p)\to S(p)$, $J(\omega)\to P(\omega)$, $H(\omega)\to -T(\omega)$, $A(t)\to B(t)$, $G(\tau)\to\hat{G}(\tau)$, and $g(s)\to\hat{g}(s)$.

There are a number of new results in Table I. In principle, any one of the quantities of a given row may be selected, and any or all of the others obtained by carrying out the proper direct or inverse integral transforms. For an arbitrary function, many of the pertinent transforms will, however, be impossible to carry out. Those cases where all or most of the positions of a row may be filled with functions expressible in closed form should be cherished, as Campbell and Foster have pointed out for the $S(i\omega)$, $B(t)$ connection.[89] By summarizing and codifying all the transform relations which obtain between the various quantities of the table, we have available Hilbert, Fourier cosine, Fourier sine, Fourier exponential, Laplace, Stieltjes, and Mellin transform tables to help in filling in the positions of a row after one of its elements is selected. In addition, direct integration, real or complex, is often helpful, as are the tables of Bierens de Haan.[44] Finally, the various direct Stieltjes inverse relations such as (66′) may often be useful.

For example, row 4 was started by taking $J(\omega)$ and $H(\omega)$ from the table of Hilbert transforms. Similarly the $J(\omega)$ and $H(\omega)$ of row 15 were obtained from Bierens de Haan's table. The $J(\omega)$ and $H(\omega)$ of No. 5 were first obtained from the Mellin transform table by noting that the necessary relation $j(s)=[\mathrm{ctn}(\pi s/2)]h(s)$ was satisfied. The $G(\tau)$ of 11 was selected as an interesting choice when it was discovered that it was listed in the table of Stieltjes transforms.

Rows 10, 11, 15, 16, 18, 19, and 21 are believed to be wholly or partly new. The $Q(p)$ and $A(t)$ connections for the others are to be found in the Fourier integral tables of Campbell and Foster.[1] These tables are based on the Fourier integral connection exemplified by Eq. (43) and its inverse, with $i\omega$ taken as p. Impulse and discontinuous functions are handled by a limiting process which is explicitly by-passed in the corresponding Laplace transforms. These tables are extremely extensive and give many hundreds of useful $Q(p)$ and $A(t)$ (or $S(p)$ and $B(t)$) pairs.

There is an apparent disagreement between the results of row 16 and the transform of the same $Q(p)$ given in the Campbell-Foster tables (No. 632). The difference arises from the fact that our single-sided Laplace transform yields an $A(t)$ defined only for $t\geqslant 0$. On the other hand, the exponential Fourier transform used in the Campbell-Foster tables is two sided and for the present $Q(p)$ is used to obtain a transform applicable in the different range $-\infty\leq t\leq\infty$. The Kronig-Kramers relations and many of our other transforms do not, therefore, apply to such functions as they do for all the functions given in Table I.

It is worth pointing out that a roundabout path through one or more different integral transforms can often give new transforms which might be difficult to obtain directly. For example, the $g(s)$ values of rows 5, 7, and 10 are new Mellin transforms not listed in the tables. In these cases, $G(\tau)$ and $g(s)$ were obtained by calculating $G(\tau)$ from $A(t)$, $a(s)$ from $A(t)$, and $g(s)$ from $a(s)$. Further, the transform method is often the simplest method of separating a complicated $Q(p)$, such as that of row 16, into its real and imaginary parts in the limit $p\rightarrow i\omega$.

The $G(\tau)$ and $A(t)$ of row 4 are of particular interest. Note that $G(\tau)$ is not normalized since $J(\omega)$ cannot be normalized at $\omega=0$. In measurements on dielectrics, it is often found that the current which flows when a constant voltage is applied or removed is proportional to $t^{-\nu}$ for long periods of time with the exponent near or equal to unity.[10,90–92] This current is just the indicial admittance $A(t)$, and it will be seen that the $A(t)$ of row 4 is of this form. The corresponding distribution function is proportional to $\tau^{-\nu}$. Although it is physically obvious that a material cannot be characterized by a distribution of relaxation times of arbitrarily increasing weight or density as τ approaches zero, this distribution may nevertheless still be an excellent approximation over a wide range of t and τ. Similar remarks apply to the $G(\tau)$ of row 18 which is also unnormalized.

Recently, a method of making measurements on dielectric materials has been described in which the system response to a linearly rising applied voltage is measured.[93] Such a ramp voltage is proportional to the integral of $u_0(t)$ which may be denoted $u_{-1}(t)=t$ $(t\geqslant 0)$. It is of interest to calculate the response of the system characterized by row 4 of Table I to such excitation, taken as $f(t)=at$. Either Eq. (2) may be used or the inverse Laplace transform of Eq. (3) carried out using the results of row 4. Either procedure gives as the system ramp response, $r(t)=2a\Gamma(\nu)\cos(\pi\nu/2)(t/\tau_0)^{1-\nu}/\pi(1-\nu)$. It is clear from this result that such a measurement on such a system would afford a sensitive method of determining ν when this quantity is close to unity. On carrying out the limit $\nu\rightarrow 1$, the above result becomes just $r(t)=a$, a constant and the first derivative of $f(t)$. Note that the $\tau^{-\nu}$ relaxation distribution function we have been considering here may be related to the excess or flicker (frequency)$^{-1}$-type noise observed in semiconductors and vacuum tubes.[94]

The $G(\tau)$ of row 21 is particularly simple since it specifies a constant density of relaxation times between τ_1 and τ_2 and none outside this region. The $G(\tau)$ of row 16 is of interest since it has an essential singularity at $\tau=0$ and oscillates in sign as τ increases. To approximate this $G(\tau)$ by means of a large number of discrete relaxation times, it would be necessary to use active elements to obtain the negative relaxation times required by negative values of $G(\tau)$. This result shows that a $J(\omega)$ of the form $\exp[-|\omega|\tau_0]$ can never be obtained or even well approximated by means of an additive combination of even an infinite number of passive RC branches of different time constants. On the other hand, all completely positive distribution functions, either continuous or discontinuous, may be approximated by a finite number of such branches; as the number of branches is increased, the response of the resulting system will approximate closer and closer to that of the system associated with the exact distribution function.

As a final example of the usefulness of transform methods in treating physical problems, it is of interest to mention that an application of (generalized) Kronig-Kramers dispersion relations[95] to the problem of pion-nucleon scattering has recently resolved, on the basis of causality, a choice between several competing assumptions which had been suggested as possibilities in the theoretical treatment of this interaction.[96]

In the present work, we have approached p from the

Table I.

No.	$Q(p)$	$J(\omega)$	$H(\omega)$	$A(t)$	$G(\tau)$	$g(s)$
1	p^{-1}	$\pi\delta(\omega)$	ω^{-1}	$u_0(t)$	$D(\lambda)=\delta(\lambda)$	$\cdots$
2	1	1	0	$\delta(t)$	$\delta(\tau)$	$\cdots$
3	p	0	$-\omega$	$\delta'(t)$	$\cdots$	$\cdots$
4	$(p\tau_0)^{\nu-1}\csc\frac{\pi\nu}{2}$ $\mathrm{Re}[p]>0,\ \ \nu-1<0$	$\lvert\omega\tau_0\rvert^{\nu-1}$ $-1<\nu-1<0$	$\mathrm{sgn}\omega\lvert\omega\tau_0\rvert^{\nu-1}\operatorname{ctn}\frac{\nu\pi}{2}$ $-1<\nu-1<0$	$\frac{2\Gamma(\nu)}{\pi\tau_0}\cos\frac{\pi\nu}{2}(t/\tau_0)^{-\nu}$ $0<\nu<1$	$\frac{2}{\pi\tau_0}\cos\frac{\pi\nu}{2}(\tau/\tau_0)^{-\nu}$ $0<\nu<1$	$\cdots$
5	$[1+p\tau_0]^{-\frac{1}{2}}$	$\frac{\cosh[\frac{1}{2}\sinh^{-1}(\omega\tau_0)]}{[1+(\omega\tau_0)^2]^{\frac{1}{4}}}$	$\frac{\sinh[\frac{1}{2}\sinh^{-1}(\omega\tau_0)]}{[1+(\omega\tau_0)^2]^{\frac{1}{4}}}$	$\frac{1}{\tau_0}\left[\frac{\tau_0}{\pi t}\right]^{\frac{1}{2}}e^{-t/\tau_0}$	$\frac{1}{\pi\tau_0}\{(\tau/\tau_0)(1-\tau/\tau_0)\}^{-\frac{1}{2}}$ $\tau/\tau_0<1$ 0 $\tau/\tau_0>1$	$\sqrt{\frac{1}{\pi}}\tau_0^{s-1}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}$ $\mathrm{Re}[s]>0$
6	$[1+p\tau_0]^{-1}$	$[1+(\omega\tau_0)^2]^{-1}$	$\frac{\omega\tau_0}{1+(\omega\tau_0)^2}$	$\frac{1}{\tau_0}e^{-t/\tau_0}$	$\delta(\tau-\tau_0)$	τ_0^{s-1}
7	$[1+p\tau_0]^{-\frac{3}{2}}$	$\frac{\cos[\frac{3}{2}\tan^{-1}\omega\tau_0]}{[1+(\omega\tau_0)^2]^{\frac{3}{4}}}$	$\frac{\sin[\frac{3}{2}\tan^{-1}\omega\tau_0]}{[1+(\omega\tau_0)^2]^{\frac{3}{4}}}$	$\frac{1}{\tau_0}\left[\frac{4t}{\pi\tau_0}\right]^{\frac{1}{2}}e^{-t/\tau_0}$	$\frac{1}{\pi\tau_0}\frac{(\tau_0/\tau-1)^{-\frac{1}{2}}}{(1-\tau/\tau_0)}$ $\tau/\tau_0<1$ 0 $\tau/\tau_0>1$	$\sqrt{\frac{4}{\pi}}\tau_0^{s-1}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)}$ $\mathrm{Re}[s]>0$
8	$[1+p\tau_0]^{-2}$	$\frac{1-(\omega\tau_0)^2}{[1+(\omega\tau_0)^2]^2}$	$\frac{2\omega\tau_0}{[1+(\omega\tau_0)^2]^2}$	$\frac{1}{\tau_0}\left(\frac{t}{\tau_0}\right)e^{-t/\tau_0}$	$-\tau\frac{d}{d\tau}[\delta(\tau-\tau_0)]$	$\tau_0^{s-1}\cdot s$
9	$\frac{p\tau_0}{1+p\tau_0}$	$\frac{(\omega\tau_0)^2}{1+(\omega\tau_0)^2}$	$\frac{-\omega\tau_0}{1+(\omega\tau_0)^2}$	$\delta(t)-\frac{1}{\tau_0}e^{-t/\tau_0}$	$\delta(\tau)-\delta(\tau-\tau_0)$	$\cdots$
10	$\frac{2\cos^{-1}(p\tau_0)}{\pi[1-(p\tau_0)^2]^{\frac{1}{2}}}$ $p\tau_0\leqslant 1$ $\frac{2\ln[p\tau_0+\{(p\tau_0)^2-1\}^{\frac{1}{2}}]}{\pi[(p\tau_0)^2-1]^{\frac{1}{2}}}$ $p\tau_0\geqslant 1$	$[1+(\omega\tau_0)^2]^{-\frac{1}{2}}$	$\frac{2\,\mathrm{sgn}\omega\ln[\lvert\omega\tau_0\rvert+\{1+(\omega\tau_0)^2\}^{\frac{1}{2}}]}{\pi[1+(\omega\tau_0)^2]^{\frac{1}{2}}}$	$\frac{2}{\pi\tau_0}K_0(t/\tau_0)$	$\frac{2}{\pi\tau_0[1-(\tau/\tau_0)^2]^{\frac{1}{2}}}$ $\tau/\tau_0<1$ 0 $\tau/\tau_0>1$	$\frac{2^{s-1}}{\pi}\tau_0^{s-1}\frac{[\Gamma(\frac{1}{2}s)]^2}{\Gamma(s)}$ $\mathrm{Re}[s]>0$
11	$\frac{1+\frac{p\tau_0}{\pi}\ln(p\tau_0)^2}{1+(p\tau_0)^2}$	$[1+\lvert\omega\tau_0\rvert]^{-1}$	$\frac{(\omega\tau_0)\ln(\omega\tau_0)^2}{\pi[(\omega\tau_0)^2-1]}$	$\frac{2}{\pi\tau_0}[\mathrm{ci}(t/\tau_0)\cos(t/\tau_0)$ $-\mathrm{si}(t/\tau_0)\sin(t/\tau_0)]$	$\frac{2}{\pi\tau_0[1+(\tau/\tau_0)^2]}$	$\tau_0^{s-1}\csc\frac{\pi s}{2}$ $0<\mathrm{Re}[s]<2$
12	$[1+(p\tau_0)^2]^{-\frac{1}{2}}$	$[1-(\omega\tau_0)^2]^{-\frac{1}{2}}$ $0<\omega\tau_0<1$ ∞ $\omega\tau_0=1$ 0 $1<\omega\tau_0<\infty$	0 $0<\omega\tau_0<1$ ∞ $\omega\tau_0=1$ $\mathrm{sgn}\omega[(\omega\tau_0)^2-1]^{-\frac{1}{2}}$ $1<\omega\tau_0<\infty$	$\frac{1}{\tau_0}J_0(t/\tau_0)$	$\cdots$	$(2\tau_0)^{s-1}\frac{\Gamma(s/2)}{\Gamma(s)\Gamma\left(1-\frac{s}{2}\right)}$ $0<\mathrm{Re}[s]<\frac{3}{2}$

TABLE I.—*Continued.*

No.	$Q(p)$	$J(\omega)$	$H(\omega)$	$A(t)$	$G(\tau)$	$g(s)$
13	$e^{-p\tau_0}$	$\cos\omega\tau_0$	$\sin\omega\tau_0$	$\delta(t-\tau_0)$	$\cdots$	$\tau_0^{s-1}[\Gamma(s)]^{-1}$
14	$\dfrac{(1-e^{-p\tau_0})}{p\tau_0}$	$\dfrac{\sin\omega\tau_0}{\omega\tau_0}$	$\dfrac{(1-\cos\omega\tau_0)}{\omega\tau_0}$	$\dfrac{1}{\tau_0}[1-u_0(t-\tau_0)]$	$\cdots$	$\tau_0^{s-1}[\Gamma(1+s)]^{-1}$
15	$\dfrac{1}{\alpha}e^{-\alpha p\tau_0}\dfrac{\sinh\alpha p\tau_0}{\sinh p\tau_0}$	$\dfrac{1}{2\alpha}\dfrac{\sin(2\alpha\omega\tau_0)}{\sin(\omega\tau_0)}$	$\dfrac{1}{\alpha}\dfrac{\sin^2(\alpha\omega\tau_0)}{\sin(\omega\tau_0)}$		$\cdots$	
16	$\dfrac{-2}{\pi}[\mathrm{ci}(p\tau_0)\sin p\tau_0 + \mathrm{si}(p\tau_0)\cos p\tau_0]$	$e^{-\lvert\omega\tau_0\rvert}$	$\dfrac{\mathrm{sgn}\,\omega}{\pi}[e^{-\lvert\omega\rvert\tau_0}\overline{Ei}(\lvert\omega\rvert\tau_0) - e^{\lvert\omega\rvert\tau_0}Ei(-\lvert\omega\rvert\tau_0)]$	$\dfrac{2}{\pi\tau_0[1+(t/\tau_0)^2]}$	$\dfrac{2}{\pi\tau}\sin(\tau_0/\tau)$	$\tau_0^{s-1}\dfrac{\csc\dfrac{\pi s}{2}}{\Gamma(s)}$ $0<\mathrm{Re}[s]<2$
17	$\dfrac{-e^{1/p\tau_0}Ei\left(-\dfrac{1}{p\tau_0}\right)}{p\tau_0}$	$\dfrac{-1}{\lvert\omega\rvert\tau_0}\left[\mathrm{ci}\left(\dfrac{1}{\lvert\omega\rvert\tau_0}\right)\sin\left(\dfrac{1}{\lvert\omega\rvert\tau_0}\right) + \mathrm{si}\left(\dfrac{1}{\lvert\omega\rvert\tau_0}\right)\cos\left(\dfrac{1}{\lvert\omega\rvert\tau_0}\right)\right]$	$\dfrac{1}{\omega\tau_0}\left[\mathrm{ci}\left(\dfrac{1}{\lvert\omega\rvert\tau_0}\right)\cos\left(\dfrac{1}{\lvert\omega\rvert\tau_0}\right) - \mathrm{si}\left(\dfrac{1}{\lvert\omega\rvert\tau_0}\right)\sin\left(\dfrac{1}{\lvert\omega\rvert\tau_0}\right)\right]$	$\dfrac{2}{\tau_0}K_0[2\sqrt{(t/\tau_0)}]$	$\tau_0^{-1}e^{-\tau/\tau_0}$	$\Gamma(s)\tau_0^{s-1}$ $\mathrm{Re}[s]>0$
18	$-e^{p\tau_0}Ei(-p\tau_0)$	$\mathrm{ci}(\lvert\omega\rvert\tau_0)\cos\omega\tau_0 - \mathrm{si}(\lvert\omega\rvert\tau_0)\sin(\lvert\omega\rvert\tau_0)$	$-\mathrm{sgn}\,\omega[\mathrm{ci}(\lvert\omega\rvert\tau_0)\sin(\lvert\omega\rvert\tau_0) + \mathrm{si}(\lvert\omega\rvert\tau_0)\cos\omega\tau_0]$	$\dfrac{1}{\tau_0[1+t/\tau_0]}$	$\dfrac{1}{\tau}e^{-\tau_0/\tau}$	$\tau_0^{s-1}\Gamma(1-s)$ $0<\mathrm{Re}\,s<1$
19	$\dfrac{1}{\pi}[e^{-p\tau_0}\overline{Ei}(p\tau_0) - e^{p\tau_0}Ei(-p\tau_0)]$	$\sin(\lvert\omega\rvert\tau_0)$	$\dfrac{-2\,\mathrm{sgn}\,\omega}{\pi}\left[\mathrm{ci}(\lvert\omega\rvert\tau_0)\sin(\lvert\omega\rvert\tau_0) + \cos\omega\tau_0\left\{\dfrac{\pi}{2}+\mathrm{si}(\lvert\omega\rvert\tau_0)\right\}\right]$	$\dfrac{2}{\pi\tau_0[1-(t/\tau_0)^2]}$	$\cdots$	$\dfrac{\tau_0^{s-1}\,\mathrm{ctn}\dfrac{\pi s}{2}}{\Gamma(s)}$ $0<\mathrm{Re}[s]<2$
20	$\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{1}{p\tau_0}\right)$	$u_0(\omega\tau_0)-u_0(\omega\tau_0-1)$	$\dfrac{1}{\pi}\ln\left\lvert\dfrac{1+\omega\tau_0}{\omega\tau_0-1}\right\rvert$	$\dfrac{2}{\pi t}\sin(t/\tau_0)$	$\cdots$	$\dfrac{2}{\pi}\tau_0^{s-1}\dfrac{\cos\dfrac{\pi s}{2}}{1-s}$ $0<\mathrm{Re}\,s<2$
21	$\dfrac{\ln\left[\dfrac{1+p\tau_2}{1+p\tau_1}\right]}{p(\tau_2-\tau_1)}$ $\tau_2>\tau_1\geqslant 0$	$\dfrac{\tan^{-1}\omega\tau_2-\tan^{-1}\omega\tau_1}{\omega(\tau_2-\tau_1)}$	$\dfrac{\ln\left[\dfrac{1+(\omega\tau_2)^2}{1+(\omega\tau_1)^2}\right]}{2\omega[\tau_2-\tau_1]}$	$\dfrac{Ei(-t/\tau_1)-Ei(-t/\tau_2)}{\tau_2-\tau_1}$	$\dfrac{u_0(\tau-\tau_1)-u_0(\tau-\tau_2)}{\tau_2-\tau_1}$	$\dfrac{1}{s}\left[\dfrac{\tau_2^s-\tau_1^s}{\tau_2-\tau_1}\right]$ $\mathrm{Re}[s]>0$

standpoint of its being a complex variable $\sigma+i\omega$. On the other hand, it may also be interpreted from the operational viewpoint as the operator $p=d/dt$.[34,97] This latter interpretation, which may be related to differential equations of infinite order, is discussed at length by Davis[97] and is the basis of the Heaviside operational calculus. In addition, our functions are related to potential theory. For example, in the region where $Q(p)$ is analytic, $\mathrm{Re}[Q(p)]$ and $\mathrm{Im}[Q(p)]$ may be considered to be scalar potential functions, such as the velocity potential and (negative) stream function of fluid flow. Both functions satisfy Laplace's equation and are termed harmonic or conjugate functions.

The relation between $S(p)$ or $Q(p)$ and potential theory can be exploited in a number of ways, one of the most familiar being the use of an electrolytic plotting tank. Here the analogy is with $\ln[S(p)]$, for example, and system gain and phase are then made analogous with the potential and the stream function along the $i\omega$ axis, respectively.[98] Guillemin[99] has also discussed a potential analog using dipoles or double layers from which the real part of $S(p)$ may be obtained. In this connection, the $J(\omega)$ and $H(\omega)$ of row 12 of Table I are associated with the problem of determining the potential due to a charged disk.[100]

In this work, we have not attempted to discuss in detail all the possible applications of the transform method to physical problems since many books have been devoted to various aspects of this subject. Instead, we have preferred to gather together, codify, and extend where possible the transform relations pertinent to a linear system. We have not striven for the highest mathematical rigor; in the application of the present results to physical problems such matters as the validity of inversion of order of integration in a double integral are usually easily settled in individual cases of interest. Further, the precise conditions under which the transforms hold are discussed at length in the references already cited. Our approach has, therefore, been largely pragmatic and formal. Although it may not be entirely satisfactory to mathematicians, it is our hope that the results will be of use both to physicists and to those concerned with circuit theory.

APPENDIX I. SUMMARY OF INTEGRAL TRANSFORMS

A. Laplace Transform

$$g(p)=\int_0^\infty e^{-pt}f(t)dt\equiv\mathfrak{L}_{(p)}[f(t)]\equiv\mathfrak{L}[f(t)].$$

p is the complex variable $\sigma+i\omega$.

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}g(p)e^{tp}dp\equiv\mathfrak{L}^{-1}[g(p)].$$

$$t\geqslant 0 \qquad \mathfrak{L}\mathfrak{L}^{-1}=1.$$

Here c is a constant which is greater than σ_a, where σ_a is the greatest lower bound of σ for which

$$\begin{matrix}\lim T\to\infty\\ \lim\epsilon\to 0\end{matrix}\int_\epsilon^T|f(t)|e^{-\sigma t}dt<\infty.$$

B. Generalized Fourier Cosine Transform

$$g(y)=\lim_{\sigma\to 0}\int_0^\infty f(x)e^{-\sigma x}\cos xydx \equiv\mathfrak{F}_{c(y)}[f(x)]\equiv\mathfrak{F}_c[f(x)].$$

The limiting operation denoted by $\lim\sigma\to 0$ is assumed included in the definition of the integral transform operator $\mathfrak{F}_c$.

$$f(x)=\mathfrak{F}_{c(x)}^{-1}[g(y)]=\frac{2}{\pi}\mathfrak{F}_{c(x)}[g(y)],$$

$$\mathfrak{F}_{c(y)}\mathfrak{F}_{c(x)}^{-1}=1=\mathfrak{F}_{c(x)}^{-1}\mathfrak{F}_{c(y)},$$

$$\mathfrak{F}_{c(x)}^{-1}=\frac{2}{\pi}\mathfrak{F}_{c(x)}.$$

When $f(x)$ is an impulse function of order $n=1$ (i.e., $\delta(x)$) or higher,

$$g(y)=\int_{-\infty\atop \lim\sigma\to 0}^\infty f(x)e^{-\sigma|x|}\cos xydx =\begin{cases}0 & n\geqslant 2\text{ and even}\\ (-1)^{(n-1)/2}y^{n-1} & n\geqslant 1\text{ and odd.}\end{cases}$$

When $f(x)$ is an odd-order impulse function, the inverse transform is (see Appendix III)

$$f(x)=\frac{1}{\pi}\mathfrak{F}_{c(x)}[g(y)]=\frac{1}{\pi}\int_{0\atop \lim\sigma\to 0}^\infty g(y)e^{-\sigma y}\cos xydy.$$

C. Generalized Fourier Sine Transform

$$g(y)=\lim_{\sigma\to 0}\int_0^\infty f(x)e^{-\sigma x}\sin xydx\equiv\mathfrak{F}_{s(y)}[f(x)]\equiv\mathfrak{F}_s[f(x)]$$

$$f(x)=\mathfrak{F}_{s(x)}^{-1}[g(y)]=\frac{2}{\pi}\mathfrak{F}_{s(x)}[g(y)],$$

$$\mathfrak{F}_{s(y)}\mathfrak{F}_{s(x)}^{-1}=1=\mathfrak{F}_{s(x)}^{-1}\mathfrak{F}_{s(y)},\quad \mathfrak{F}_{s(x)}^{-1}=\frac{2}{\pi}\mathfrak{F}_{s(x)}.$$

When $f(x)$ is an impulse function,

$$g(y)=\lim_{\sigma\to 0}\int_{-\infty}^\infty f(x)e^{-\sigma|x|}\sin xydx =\begin{cases}0 & n\text{ odd}\\ (-1)^{n/2}y^{n-1} & n\geqslant 0\text{ and even.}\end{cases}$$

Further, when $f(x)$ is an even-order impulse function, the inverse transform is (see Appendix III)

$$f(x)=\frac{1}{\pi}\mathfrak{F}_{s(x)}[g(y)]=\lim_{\sigma\to 0}\frac{1}{\pi}\int_0^\infty g(y)e^{-\sigma y}\sin xy dy.$$

D. Exponential Fourier Transform

$$g(y)=\int_{-\infty}^{\infty} f(x)e^{-ixy}dx\equiv\mathfrak{F}_e[f(x)],$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} g(y)e^{ixy}dy\equiv\mathfrak{F}_e^{-1}[g(y)].$$

E. Hilbert Transform

$$g(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(y)dy}{x-y},$$

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(y)dy}{y-x}.$$

The integrals are Cauchy principal values and relate the real and imaginary parts of a complex function $S(x)=g(x)+if(x)$.

F. Kronig-Kramers Transforms

$$g(x)=\frac{2}{\pi}\int_0^\infty\frac{yf(y)dy}{x^2-y^2},$$

$$f(x)=\frac{2x}{\pi}\int_0^\infty\frac{g(y)dy}{y^2-x^2}.$$

The integrals are Cauchy principal values and relate the real and imaginary parts of a complex function $S(x)=g(x)+if(x)$. Here the real part is even in x, the imaginary part odd in x. Further, $S(\infty)$ and $[S(x)/x]_\infty$ are taken zero for simplicity.

G. Stieltjes Transform

$$g(y)=\int_0^\infty\frac{f(x)dx}{x+y}\equiv\mathcal{S}_{(y)}[f(x)]\equiv\mathcal{S}[f(x)]\equiv\mathcal{L}_{(y)}\mathcal{L}_{(u)}[f(x)],$$

$$f(x)=\lim_{\epsilon\to 0}\frac{1}{\pi}\operatorname{Im}[g(-x-i\epsilon)].$$

H. Mellin Transform

$$g(s)=\int_0^\infty f(x)x^{s-1}dx\equiv\mathfrak{M}[f(x)].$$

s is a complex variable $\epsilon+i\alpha$.

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-s}g(s)ds\equiv\mathfrak{M}^{-1}[g(s)].$$

Here c is a constant greater than ϵ_a, where ϵ_a is the greatest lower bound of ϵ for which

$$\begin{matrix}\lim T\to\infty\\ \lim\delta\to 0\end{matrix}\int_\delta^T x^{\epsilon-1}|f(x)|dx<\infty.$$

We have not given detailed conditions for the validity of all the above transforms. These matters are dealt with at length in, e.g., references 4, 14, 25, 45, 53, 63, and 75. For our purposes, it is generally sufficient that the integrals involved converge absolutely to a finite value.

APPENDIX II. IMPULSE FUNCTION RELATIONS

The unit impulse functions of nth order, $u_n(x)$ are usually known to physicists as the unit step the unit impulse or Dirac delta function, the unit doublet, etc., for $n=0, 1, 2, \cdots$, respectively. We adhere to these conventions in the present work by designating these functions by the symbols $u_0(x)$, $\delta(x)$, $\delta'(x)$, etc. This appendix comprises a collection of useful representations of these functions and relations which they obey.

$$u_0(x)=\int_{-\infty}^{\infty}\delta(x)dx=1\quad(x>0)$$

$$\int_{-\infty}^{\infty}x\delta(x)dx=0\quad x\delta(x)=0$$

$$\int_{-\infty}^{\infty}f(x)\delta(x)dx=f(0)$$

(except when $f(x)$ is not regular at $x=0$)

$$\int_{-\infty}^{\infty}f(x)\delta(x-a)dx=f(a)$$

(except when $f(x)$ is not regular at $x=a$)

$$\delta(-x)=\delta(x)\qquad\delta(ax)=a^{-1}\delta(x)\quad(a>0)$$

$$\delta'(-x)=-\delta'(x)\quad x\delta'(x)=-\delta(x)$$

$$f(x)\delta'(x)=f(0)\delta'(x)-f'(x)\delta(x)$$

(except when $f(x)$ is not regular at $x=0$)

$$\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\pm i\alpha x}d\alpha$$

$$\delta(x)=\frac{2}{\pi^2}\int_0^\infty\frac{dy}{y^2-x^2}$$

$$\delta(x)=\lim_{\alpha\to 0}\frac{1}{\pi}\left[\frac{\alpha}{\alpha^2+x^2}\right]$$

$$\delta(x-a)=\lim_{\alpha\to 0}\frac{1}{\pi}\left[\frac{\alpha}{\alpha^2+(x-a)^2}\right]$$

$$\delta(x)=\lim_{\alpha\to 0}\frac{1}{\pi}\left[\frac{\sin\alpha x}{x}\right]$$

$$\delta(x)=\lim_{\alpha\to 0}\left[\frac{e^{-x^2/4k\alpha}}{(4\pi k\alpha)^{\frac{1}{2}}}\right]\quad(k>0)$$

$$\delta(x)=\lim_{\lambda\to\infty}\left[\frac{\lambda}{\sqrt{\pi}}e^{-\lambda^2x^2}\right]$$

$$\delta'(x)=\frac{d}{dx}[\delta(x)]=\frac{\pm i}{2\pi}\int_{-\infty}^{\infty}\alpha e^{\pm i\alpha x}d\alpha$$

$$\delta'(x)=\lim_{\alpha\to 0}\frac{-2}{\pi}\left[\frac{x\alpha}{(\alpha^2+x^2)^2}\right]$$

$$\delta^{(n)}(x)=\lim_{\alpha\to 0}\frac{(-1)^n}{\pi}n!\frac{\sin\{(n+1)\tan^{-1}(\alpha/x)\}}{(\alpha^2+x^2)^{\frac{1}{2}(n+1)}}$$

$$\delta(x)=\lim_{\alpha\to 0}\frac{1}{\pi}\int_0^{\infty}e^{-\alpha t}\cos xt dt$$
$$=\frac{1}{\pi}\mathfrak{F}_c[u_0(t)]=\lim_{\alpha\to 0}\frac{1}{\pi}\mathfrak{L}_{(\alpha)}[\cos xt]$$

$$\delta'(x)=\lim_{\alpha\to 0}\frac{-1}{\pi}\int_0^{\infty}te^{-\alpha t}\sin xt dt$$
$$=-\frac{1}{\pi}\mathfrak{F}_s[t]=\lim_{\alpha\to 0}\frac{-1}{\pi}\mathfrak{L}_{(\alpha)}[t\sin xt]$$

$$\delta(x^2-a^2)=(2a)^{-1}[\delta(x-a)+\delta(x+a)]\quad(a>0)$$

$$\delta(x^{-1}-y^{-1})=xy\delta(x-y)=x^2\delta(x-y)=y^2\delta(x-y)$$

$$\delta(x-b)=\frac{1}{\pi}\frac{f(x)\delta(x-a)}{(x-b)^2+[\delta(x-a)f(x)]^2}$$

$$\delta[h(x)]=\lim_{\alpha\to 0}\frac{1}{\pi}\left[\frac{\alpha}{\alpha^2+[h(x)]^2}\right]$$
$$=\frac{1}{\pi}\frac{f(x)\delta(x-a)}{[h(x)]^2+[f(x)\delta(x-a)]^2}=\sum_n\frac{\delta(x-x_n)}{|h'(x_n)|},$$

where $h(x_n)=0$, $h(a)\neq 0$. These last two relations were first given by Gross and Pelzer.[81]§

$$\mathfrak{L}[u_0(x)]=p^{-1}$$
$$\mathfrak{F}_c[u_0(x)]=\pi\delta(y)$$
$$\mathfrak{F}_s[u_0(x)]=y^{-1}$$
$$\mathfrak{L}[\delta(x)]=1\qquad \mathfrak{L}[\delta'(x)]=p$$
$$\mathfrak{L}[\delta(x-a)]=e^{-ap}\qquad \mathfrak{F}_c[\delta'(x)]=0$$

§ *Note added in proof.*—B. Gross, Lineare Systeme, Supplement to Nuovo cimento **3**, 235 (1956), has pointed out (p. 292) that this expansion for $\delta[h(x)]$ applies only for simple zeros of $h(x)$. An expansion involving both $\delta(x-x_n)$ and $\delta'(x-x_n)$ is necessary when higher order zeros are present.

$$\mathfrak{F}_c[\delta(x)]=1\qquad \mathfrak{F}_s[\delta'(x)]=-y$$
$$\mathfrak{F}_s[\delta(x)]=0\qquad \mathcal{S}[\delta'(x)]=y^{-2}.$$
$$\mathcal{S}[\delta(x)]=y^{-1}$$
$$\mathfrak{M}[\delta(x)]=0\quad(s\geqslant 1)$$

In passing from Eq. (14) to Eqs. (17) and (18) of the text, a partial rationalization was carried out which led eventually to the useful forms (17′) and (18′) for $J(\omega)$ and $H(\omega)$. For all rational system functions which are nonsingular at the origin, we believe such rationalization is valid. This class includes positive real functions as a subclass. In addition, if $S(p)$ is analytic in the right-half plane (with the possible exception of the point at infinity), (17′), (18′), (17″), and (18″) will hold. Hence, it is apparent that these equations are applicable to a large class of important functions.

Here, we are interested in the form of $S(p)$ necessary for the above rationalization to be invalid. In general, we may state that it will be invalid whenever $S(p)$ has a denominator whose real or imaginary part is proportional to σ, for σ an infinitesimal. In this case, either $\mathrm{Re}[S(\sigma+i\omega)]$ or $\mathrm{Im}[S(\sigma+i\omega)]$ will be of the form $\sigma f(\omega)/[g(\omega)\sigma^2+h(\omega)]$ where $f(\omega)$, $g(\omega)$, and $h(\omega)$ are functions of ω or constants. Then in either (17) or (18) will appear products of terms whose individual limits ($\sigma\to 0$) would be delta functions. Such products are difficult to interpret and are meaningless without further interpretation. They may be avoided in individual cases by multiplying out the denominator of (14) as it stands, rationalizing, and finally taking the limit $\sigma\to 0$. An example where this process is necessary is the $D(\lambda)$ of Eq. (96).

Another example of interest is that where $S(p)=p^{-1}$. Then,

$$Q(i\omega)=\lim_{\sigma\to 0}[\sigma+i\omega]^{-2}\neq\lim_{\sigma\to 0}\left[\frac{\sigma-i\omega}{\sigma^2+\omega^2}\right]^2$$
$$=\lim_{\sigma\to 0}[(\sigma^2-\omega^2)+2i\omega\sigma]^{-1}=\lim_{\sigma\to 0}\left[\frac{(\sigma^2-\omega^2)-2i\omega\sigma}{\omega^4+2\sigma^2\omega^2+\sigma^4}\right]$$
$$=-\omega^{-2}-\lim_{\sigma\to 0}i\left[\frac{2\omega\sigma}{(\omega^2+\sigma^2)^2}\right]=-\omega^{-2}+i\pi\delta'(\omega).$$

Note that ω^{-2} in the above is approached as a limit from the right and should actually be denoted $(\omega^{-2})_+$. In evaluating $Q(i\omega)$ we have taken

$$\lim_{\sigma\to 0}\left[\frac{\sigma^2}{(\sigma^2+\omega^2)^2}\right]=\lim_{\sigma\to 0}\left[\frac{\sigma}{\sigma^2+\omega^2}\right]^2=0$$
$$\neq\left[\lim_{\sigma\to 0}\frac{\sigma}{\sigma^2+\omega^2}\right]^2=[\pi\delta(\omega)]^2.$$

We shall discuss products of impulse functions and of impulse functions and functions singular at the origin below.

Other functions for which the rationalization is invalid are $S(p)=\text{sech}\,ap$, $S(p)=\text{csch}\,ap$, etc. In obtaining $Q(i\omega)$ directly from (14) for such functions, it is generally necessary to neglect higher order terms involving σ^2 before rationalizing as was done, e.g., in passing from (96) to (96′). Functions of this type lead to $J(\omega)$ and $H(\omega)$'s which may involve terms such as $\delta[f(\omega)]$ and hence may be expressed, in general, as series of terms involving $\delta(\omega-\omega_n)$, $n=0,1,2,\cdots$. The neglect of higher order terms in σ will not lead to ambiguities unless such neglect introduces a possible impulse function at the origin as it would in the case $S(p)=p^{-1}$. There, if we neglect σ^2 when it first appears, we obtain

$$Q(i\omega)=\lim_{\sigma\to 0}[-\omega^2+2i\omega\sigma]^{-1}=\lim_{\sigma\to 0}\left[\frac{-\omega^2-2i\omega\sigma}{\omega^4+4\omega^2\sigma^2}\right]$$

$$=-\omega^{-2}-\lim_{\sigma\to 0} i\left[\frac{2\omega\sigma}{\omega^4+4\omega^2\sigma^2}\right].$$

To evaluate the remaining limit, we must divide through by $4\omega^2$, but such division introduces a possible additional term $b\delta(\omega)$ where b is a constant. Therefore,

$$H(\omega)=\lim_{\sigma\to 0}\left[\frac{1}{2\omega}\frac{\sigma}{\sigma^2+(\omega/2)^2}\right]+b\delta(\omega)$$

$$=\pi\omega^{-1}\delta(\omega)+b\delta(\omega).$$

It is by no means clear that this result is either correct or equals the previous $H(\omega)=-\pi\delta'(\omega)$; however, some light will be shed on the matter by the following considerations.

The question of products of impulse functions and of impulse functions and functions singular at the origin may be treated by means of the distribution theory of Schwartz.[53,54]‖ These matters have been investigated by this means by Güttinger,[56] whose results we shall summarize and extend for our present purposes as follows. First, products of the above types are generally noncommutative and nonassociative. For such products we shall use Güttinger's "o" symbol in place of the usual multiplicative dot. Functions such as x^{-1} must be considered in terms of their principal values when they appear in products like $x^{-1}\circ\delta(x)$; we shall omit explicit reference to this fact and let it be understood. In addition, for the following formulas, the argument of the impulse functions will always be x and will be omitted. The constants c and c' below are arbitrary, finite, complex constants. The prime denotes differentiation with respect to the argument.

Now if $f(x)$ is an indefinitely differentiable function and A and B are improper operators, or distributions such as $\delta(x)$ or x^{-1}, then

$$f\circ A=fA=Af,$$

$$f\circ(A\circ B)=A\circ(Bf)=A\circ(fB).$$

We may also require that the distribution law of differentiation of an ordinary product applies to the present class of products, so that

$$(A\circ B)'=A\circ B'+A'\circ B.$$

The following product relations then hold[101]:

$$\delta^{(m)}\circ u_0=-\sum_{r=0}^{m}c_r\delta^{(m-r)}$$

$$\delta^{(m)}\circ\delta=c_{m+1}\delta$$

$$\delta^{(k)}\circ\delta^n=c_{n+1,\,n}\delta$$

$$x\circ\delta^n=x\delta^n=0$$

$$u_0\circ u_0=u_0^2=u_0$$

$$u_0\circ\delta=(1+c_0)\delta$$

$$\delta\circ u_0=-c_0\delta$$

$$u_0'=\delta$$

$$\delta\circ\delta=\delta^2=c_1\delta$$

$$\delta\circ\delta'=c_1\delta'-c_2\delta$$

$$\delta'\circ\delta=c_2\delta$$

$$x^{-1}\circ\delta=c_0'\delta$$

$$\delta\circ x^{-1}=-\delta'+c_0'\delta$$

$$x^{-1}\circ\delta'=c_0'\delta'+c_1'\delta$$

$$x^{-2}\circ\delta'=c_1'\delta'+2c_2'\delta$$

$$\delta'\circ x^{-1}=-\tfrac{1}{2}\delta''+c_1'\delta$$

$$x^{-2}\circ\delta=c_1'\delta$$

$$\delta\circ x^{-2}=\tfrac{1}{2}\delta''-c_0'\delta'+c_1'\delta$$

$$x^{-1}\circ\delta''=c_0'\delta''+2c_1'\delta'+2c_2'\delta$$

$$x^{-1}\circ\delta'''=c_0'\delta'''+3c_1'\delta''+6c_2'\delta'+6c_3'\delta$$

$$x\circ\delta'=\delta'\circ x=x\delta'=-\delta$$

$$x\circ\delta''=\delta''\circ x=x\delta''=-2\delta'$$

$$x\circ(x^{-1}\circ\delta)=0$$

$$x\circ(\delta\circ x^{-1})=\delta$$

$$x\circ(x^{-1}\circ\delta')=-c_0'\delta$$

$$x\circ(\delta'\circ x^{-1})=\delta'$$

$$x^{-1}\circ x^{-1}=x^{-2}+c\delta$$

$$x^{-1}\circ x^{-2}=x^{-3}-c\delta'$$

$$x^{-2}\circ x^{-1}=x^{-3}.$$

The c and c' constants are not equal in general. A scrutiny of the parity of these equations suggests that c_{2n} and c_{2n}' can be taken zero without changing the

‖ *Note added in proof.*—A different method of handling such products has been developed by B. Gross, Lineare Systeme, Supplement to Nuovo cimento 3, 235 (1956).

result of integrating the product operators from $-\infty$ to $+\infty$. For example, adding $\delta'(x) \circ \delta(x)$ and the same quantity with x negative yields $\delta'(x) \circ \delta(x) - \delta'(x) \circ \delta(x) = 0 = 2c_2\delta(x)$. Hence, $c_2 = 0$.

It is now clear that the second calculation of $H(\omega)$ above in which σ^2 was neglected yields an ambiguous result since we do not know whether the $\omega^{-1}\delta(\omega)$ term is really $\omega^{-1} \circ \delta(\omega)$ or $\delta(\omega) \circ \omega^{-1}$. Making the latter choice and using the above expression for the product we find

$$H(\omega) = -\pi\delta'(\omega) - c_0'\pi\delta(\omega) + b\delta(\omega) = -\pi\delta'(\omega),$$

since the even terms on the right cancel because $H(\omega)$ must be odd. For the same reason, $\delta(\omega)$ terms in $H(\omega)$ cannot contribute anything in any of the integrals involving this quantity.

As a further indication that $H(\omega)$ really is $-\pi\delta'(\omega)$ and $J(\omega) = \omega^{-2}$ for $Q(p) = p^{-2}$, we may note that $A(t) = t$. The Fourier sine and cosine transforms of $A(t)$ lead to the above $H(\omega)$ and $J(\omega)$. In addition, this value of $A(t)$ may be recovered from either $J(\omega)$ or $H(\omega)$ by integration by parts. For example,

$$A(t) = -\lim_{\sigma\to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\sigma|\omega|}\omega^{-2} \cos(t\omega) d\omega,$$

$$A(t) = \lim_{\sigma\to 0} \frac{e^{-\sigma|\omega|}}{\pi\omega} \cos t\omega \Big|_{-\infty}^{\infty} + \frac{t}{\pi} \int_{-\infty}^{\infty} \omega^{-1} e^{-\sigma|\omega|} \sin t\omega d\omega$$

$$= \lim_{\sigma\to 0} \frac{2t}{\pi} \int_0^{\infty} e^{-\sigma\omega}\omega^{-1} \sin t\omega d\omega$$

$$= \lim_{\sigma\to 0} \frac{2t}{\pi} \tan^{-1}\frac{t}{\sigma} = t.$$

It has been mentioned earlier that if an equation of the form $f(x) = ag(x)$ holds, then, in general, we can only infer that $f(x)/g(x) = a + b\delta(x)$ in the case when in the limit $x = 0$, $f(x)/g(x) \to \infty$. If $f(x)$ is an impulse function, the quotient $f(x)/g(x)$ must also be interpreted as either $[g(x)]^{-1} \circ f(x)$ or $f(x) \circ [g(x)]^{-1}$ and the principal value taken when integrating. These considerations impose certain restrictions on $Q(p) = S(p)/p$. Since this equation is defined without the added delta function term, the quantity p cannot be identically zero. When $\omega = 0$, we must interpret $\lim \sigma \to 0$ as implying that we can approach arbitrarily close to the point $p = 0$ along the σ axis but cannot reach it; in this sense it may be considered a limit point only.

Since $S(p)$ and $Q(p)$ are uniquely related, the real and imaginary parts of $S(i\omega)$ and $Q(i\omega)$ must also be so related. As we have seen, for any specific choice of $S(p)$ or $Q(p)$ all the real and imaginary parts can be calculated. It is therefore evident that the relations between the quantities $J(\omega)$ and $P(\omega)$ and $T(\omega)$ and between $H(\omega)$ and $P(\omega)$ and $T(\omega)$ cannot involve arbitrary constants and must be unique. We have already seen that the expressions (17′) and (18′) for $J(\omega)$ and $H(\omega)$ do not lead to unique results for some functions. Here, we wish to investigate the possibility of rewriting these results in such a form that they will hold uniquely for all functions of interest. It turns out that such a revision can be accomplished by writing the products occurring in (17′) and (18′) as improper type products and making a specific choice of some of the arbitrary c_i and c_i' constants. The results are

$$J(\omega) = T(\omega) \circ \omega^{-1} + \pi P(\omega) \circ \delta(\omega),$$
$$H(\omega) = \omega^{-1} \circ P(\omega) - \pi\delta(\omega) \circ T(\omega),$$

with $c_{2n} = c_{2n}' = 0$ $(n = 0, 1, 2 \cdots)$, and $c = \pi^2 c_1$.

These formulas for $J(\omega)$ and $H(\omega)$ may be checked by direct comparison with the results obtained on expanding $Q(p) = p^{-n}$ and $S(p) = p^{1-n}$, $n = 1, 2, 3, \cdots$. For example, we have considered the case $Q(p) = p^{-3}$. To obtain the corresponding $J(\omega)$ and $H(\omega)$ directly, an expression for $\delta''(\omega)$ as a limit is required. If we differentiate the result given earlier in this appendix for $\delta'(\omega)$ or use the expression for $\delta^{(n)}(\omega)$ we find

$$\delta''(\omega) = \lim_{\alpha\to 0} \frac{2\alpha}{\pi} \frac{3\omega^2 - \alpha^2}{(\alpha^2 + \omega^2)^3}.$$

On using this result, expanding $Q(p)$, and taking the limit $\sigma \to 0$, we find $J(\omega) = -(\pi/2)\delta''(\omega)$, $H(\omega) = -\omega^{-3}$. Further, for $S(p) = p^{-2}$, $P(\omega) = -\omega^{-2}$, $T(\omega) = \pi\delta'(\omega)$. It is readily verified that the above o-product relations for $J(\omega)$ and $H(\omega)$ are consistent with these results.

APPENDIX III. INVERSION OF TWO TRANSFORM RELATIONS

We first require inversion of an equation of the form

$$F(\omega) = \int_0^{\infty} f(\tau) e^{-\sigma'\tau} \cos\omega\tau d\tau.$$

Let us multiply both sides by $e^{-\sigma\omega} \cos\omega t d\omega$ and integrate from 0 to ∞. We obtain

$$\int_0^{\infty} F(\omega) e^{-\sigma\omega} \cos\omega t d\omega = \int_0^{\infty} \left\{ e^{-\sigma\omega} \cos\omega t \int_0^{\infty} f(\tau) e^{-\sigma'\tau} \cos\omega\tau d\tau \right\} d\omega,$$

which becomes, on inverting the order of integration on the right,

$$\int_0^{\infty} F(\omega) e^{-\sigma\omega} \cos\omega t d\omega = \int_0^{\infty} f(\tau) e^{-\sigma'\tau} \left\{ \int_0^{\infty} e^{-\sigma\omega} \cos\omega t \cos\omega\tau d\omega \right\} d\tau$$

$$= \frac{1}{2} \int_0^{\infty} f(\tau) e^{-\sigma'\tau} \left[\frac{\sigma}{\sigma^2 + (\tau - t)^2} + \frac{\sigma}{\sigma^2 + (\tau + t)^2} \right] d\tau.$$

If we next carry out the limit $\sigma \to 0$ on the right, we obtain

$$\int_0^\infty F(\omega)e^{-\sigma\omega}\cos\omega t d\omega = \frac{\pi}{2}\int_0^\infty f(\tau)e^{-\sigma'\tau}[\delta(\tau-t)+\delta(\tau+t)]d\tau.$$

Assuming that $f(-\tau)$ is zero for any nonzero positive value of τ, we may extend the lower limit of integration on the right to $-\infty$; then carrying out the integration yields

$$\int_0^\infty F(\omega)e^{-\sigma\omega}\cos\omega t d\omega = \frac{\pi}{2}[f(t)+f(-t)],$$

where we have let σ' go to zero. Ordinary functions for which $f(-t)\equiv 0$ give the desired result

$$f(t) = \frac{2}{\pi}\int_0^\infty_{\lim\sigma\to 0} F(\omega)e^{-\sigma\omega}\cos\omega t d\omega = \mathfrak{F}_c^{-1}[F(\omega)] = \frac{2}{\pi}\mathfrak{F}_c[F(\omega)].$$

However, if $f(t)$ is proportional to $\delta(t)$, the relation becomes

$$f(t) = \frac{1}{\pi}\int_0^\infty_{\lim\sigma\to 0} F(\omega)e^{-\sigma\omega}\cos\omega t d\omega = \frac{1}{\pi}\mathfrak{F}_c[F(\omega)]$$

because $\delta(-t)=\delta(t)$. For $f(t)$ to equal $\delta(t)$, it is obviously necessary that $F(\omega)=1$ (see Appendix II). We shall assume that the definition of $\mathfrak{F}_c$ is extended to take care of terms in $f(t)$ proportional to $\delta(t)$ in the above manner. Then if a is a constant, we have

$$\mathfrak{F}_c[a] = a\pi\delta(t).$$

The inversion of (22) and (24) is somewhat more difficult if proper account is to be taken of the values of $S(i\omega)$ and $Q(i\omega)$ at $\omega=\infty$. These values are, of course, $P(\infty)$ and $J(\infty)$ and will often be zero.

We shall first invert (22) using an indirect method. The earlier part of this appendix establishes Eq. (27)

$$A(t) = \frac{2}{\pi}\int_0^\infty_{\lim\sigma\to 0} J(\omega)e^{-\sigma\omega}\cos\omega t d\omega.$$

On differentiating this equation with respect to t and using (5), we obtain

$$B(t) = A(0)\delta(t) - \frac{2}{\pi}\int_0^\infty_{\lim\sigma\to 0} \omega J(\omega)e^{-\sigma\omega}\sin\omega t d\omega.$$

Next, using (16'), we find

$$B(t) = A(0)\delta(t) - \frac{2}{\pi}\int_0^\infty_{\lim\sigma\to 0} T(\omega)e^{-\sigma\omega}\sin\omega t d\omega,$$

the desired result. It is easy to show from the initial and final value theorems of the Laplace transform that

$$A(0) = P(\infty),$$

and

$$A(\infty) = P(0).$$

The inversion of (24) may be carried out in a completely analogous manner to the above by going down another step in p. The above results made use of relations between the inverse Laplace transforms of $Q(p)=S(p)/p$ and $S(p)$. We may define a new function $V(p)=Q(p)/p$ and use relations between $v(t)$, the inverse transform of $V(p)$, and $A(t)$. $v(t)$ and $A(t)$ will of course be related in the same manner as $A(t)$ and $B(t)$. Since the initial value theorem shows that $v(0)=J(\infty)$, we immediately obtain (28). As in the preceding $\mathfrak{F}_c$ inversion, if the time function involves an impulse (e.g., doublet impulse) at the origin, then the factor $2/\pi$ multiplying the result must be changed to $1/\pi$. This follows from an inversion analogous to that of the first part of this appendix. Such an inversion of equations involving $\mathfrak{F}_s$ instead of $\mathfrak{F}_c$ yields

$$f(t) - f(-t) = \frac{2}{\pi}\int_0^\infty_{\lim\sigma\to 0} F(\omega)e^{-\sigma\omega}\sin\omega t d\omega.$$

We see that if $f(t)$ involves a function such as $P(\infty)\cdot\delta(t)$ which is even at the origin, this method of inversion does not yield this term, and the preceding method must be employed. Similarly, if a function odd at the origin such as $\delta'(t)$ is involved, the above equation becomes

$$f(t) = \lim_{\sigma\to 0}\frac{1}{\pi}\int_0^\infty F(\omega)e^{-\sigma\omega}\sin\omega t d\omega.$$

The inversion of an equation involving $\mathfrak{F}_c$ in the first part of this Appendix yielded

$$f(t) + f(-t) = \frac{2}{\pi}\int_0^\infty F(\omega)e^{-\sigma\omega}\cos\omega t d\omega.$$

It is necessary to point out that if $f(t)$ here involves $\delta'(t)$, the above inversion will not yield this term in $f(t)$. This observation is pertinent to Eq. (25), where we have added a term $J(\infty)\delta'(t)$ in the expression for $B(t)$. Its presence can be justified by a technique similar to that which introduced the term $A(0)\cdot\delta(t)$ in (26) from the inversion of (22).

APPENDIX IV. TRANSFORMATION OF THE KRONIG-KRAMERS RELATIONS

Equation (29) may be written as

$$P(\omega) = P(\infty) - \lim_{\sigma,\sigma'\to 0}\frac{2}{\pi}\int_0^\infty e^{-\sigma' x}\cos\omega x \times\left[\int_0^\infty T(y)e^{-\sigma y}\sin xy dy\right]dx.$$

On interchanging the order of integration, we obtain

$$P(\omega)=P(\infty)-\lim_{\sigma,\sigma'\to 0}\frac{2}{\pi}\int_0^\infty T(y)e^{-\sigma y}\times\left[\int_0^\infty e^{-\sigma' x}\cos\omega x\,\sin yx dx\right]dy$$

$$=P(\infty)-\lim_{\sigma,\sigma'\to 0}\frac{1}{\pi}\int_0^\infty T(y)e^{-\sigma y}\times\left[\frac{y+\omega}{\sigma'^2+(y+\omega)^2}+\frac{y-\omega}{\sigma'^2+(y-\omega)^2}\right]dy.$$

Now let $\sigma'\to 0$ and collect terms; then

$$P(\omega)=P(\infty)-\lim_{\sigma\to 0}\frac{2}{\pi}⨍_0^\infty \frac{yT(y)e^{-\sigma y}dy}{y^2-\omega^2},$$

where the $⨍$ sign denotes that the Cauchy principal value of the integral is indicated.

In a similar manner, Eq. (30) yields

$$T(\omega)=\lim_{\sigma,\sigma'\to 0}\frac{-2}{\pi}\int_0^\infty P(y)e^{-\sigma y}\times\left[\int_0^\infty e^{-\sigma' x}\sin\omega x\,\cos xy dx\right]dy$$

$$=\lim_{\sigma,\sigma'\to 0}\frac{-1}{\pi}\int_0^\infty P(y)e^{-\sigma y}\times\left[\frac{\omega+y}{\sigma'^2+(\omega+y)^2}+\frac{\omega-y}{\sigma'^2+(\omega-y)^2}\right]dy.$$

Again on letting $\sigma'\to 0$ and collecting terms, we obtain

$$T(\omega)=\lim_{\sigma\to 0}\frac{-2\omega}{\pi}⨍_0^\infty\frac{P(y)e^{-\sigma y}dy}{\omega^2-y^2}.$$

When $P(y)$ is proportional to $\delta(y)$, this integral may be written

$$T(\omega)=\frac{-\omega}{\pi}\int_{-\infty}^\infty\frac{P(y)dy}{\omega^2-y^2}.$$

Since the convergence factors $e^{-\sigma y}$ in the above integrals are superfluous, the limit $\sigma\to 0$ has been carried out explicitly before integration in writing these integrals in the text.

APPENDIX V. SYSTEM RELATIONS IN THE MELLIN TRANSFORM PLANE

In this appendix, we summarize some of the most useful relations between the Mellin transforms of quantities characterizing a linear system.

$$p(s)=-\operatorname{ctn}\frac{\pi s}{2}t(s)=\cos\frac{\pi s}{2}s_p(s)=\Gamma(s)\cos\frac{\pi s}{2}b(1-s), \quad (1)$$

$$t(s)=-\tan\frac{\pi s}{2}p(s)=-\sin\frac{\pi s}{2}s_p(s)=-\Gamma(s)\sin\frac{\pi s}{2}b(1-s), \quad (2)$$

$$b(s)=\frac{2}{\pi}\Gamma(s)\cos\frac{\pi s}{2}p(1-s)=\frac{-2}{\pi}\Gamma(s)\sin\frac{\pi s}{2}t(1-s)=\frac{\Gamma(s)}{\pi}\sin\pi s s_p(1-s), \quad (3)$$

$$s_p(s)=\sec\frac{\pi s}{2}p(s)=-\csc\frac{\pi s}{2}t(s)=\Gamma(s)b(1-s), \quad (4)$$

$$s_{i\omega}(s)=e^{-i\pi s/2}s_p(s), \quad (5)$$

$$b(s)=(1-s)a(s-1), \quad (6)$$

$$j(s)=\operatorname{ctn}\frac{\pi s}{2}h(s)=\cos\frac{\pi s}{2}q_p(s)=\frac{\pi}{2}\csc\frac{\pi s}{2}d(s)=\frac{\pi}{2}\csc\frac{\pi s}{2}g(1-s)=\Gamma(s)\cos\frac{\pi s}{2}a(1-s), \quad (7)$$

$$h(s)=\tan\frac{\pi s}{2}j(s)=\sin\frac{\pi s}{2}q_p(s)=\frac{\pi}{2}\sec\frac{\pi s}{2}d(s)=\frac{\pi}{2}\sec\frac{\pi s}{2}g(1-s)=\Gamma(s)\sin\frac{\pi s}{2}a(1-s), \quad (8)$$

$$a(s)=\Gamma(s)g(s)=\Gamma(s)d(1-s)=\frac{2}{\pi}\Gamma(s)\cos\frac{\pi s}{2}j(1-s)=\frac{\Gamma(s)}{\pi}\sin\pi s q_p(1-s)=\frac{2}{\pi}\Gamma(s)\sin\frac{\pi s}{2}h(1-s), \quad (9)$$

$$g(s)=[\Gamma(s)]^{-1}a(s)=d(1-s)=\frac{2}{\pi}\cos\frac{\pi s}{2}j(1-s)=\frac{1}{\pi}\sin\pi s q_p(1-s)=\frac{2}{\pi}\sin\frac{\pi s}{2}h(1-s), \quad (10)$$

$$d(s)=\frac{2}{\pi}\sin\frac{\pi s}{2}j(s)=\frac{2}{\pi}\cos\frac{\pi s}{2}h(s)=\frac{1}{\pi}\sin\pi s q_p(s)=g(1-s)=\frac{\Gamma(s)}{\pi}\sin\pi s a(1-s), \quad (11)$$

$$q_p(s) = \sec\frac{\pi s}{2} j(s) = \csc\frac{\pi s}{2} h(s) = \pi \csc\pi s d(s)$$

$$= \pi \csc\pi s g(1-s) = \Gamma(s) a(1-s), \quad (12)$$

$$q_{i\omega}(s) = e^{-i\pi s/2} q_p(s). \quad (13)$$

It is worth mentioning that an equation equivalent to the extreme left and right sides of (12) has been given by Van Der Pol and Bremmer[25] (the example at the top of p. 254) as an interesting property of electrical networks. The present appendix shows that this equation is one of many similar relations which apply in the Mellin plane for linear systems.

Note that in the above we have assumed that possible nonzero $J(\infty)$ and $P(\infty)$ terms have been eliminated from $Q(p)$ and $S(p)$ by normalization. It will be seen that the Mellin transforms of $S(p)$ and $S(i\omega)$ have been denoted by $s_p(s)$ and $s_{i\omega}(s)$ respectively; a similar convention has been adopted for the transforms of $Q(p)$ and $Q(i\omega)$. A few of the above transform relations involving $j(s)$ and $h(s)$ have been derived previously.[31] Note that the normalization condition for $G(\tau)$ is simply that $g(1)=1$ [or $g(1)=J(0)$ if $Q(i\omega)$ has not been normalized so that $J(0)=1$].

REFERENCES

1. G. A. Campbell and R. M. Foster, *Fourier Integrals for Practical Applications* (D. Van Nostrand Company, Inc., New York, 1948).
2. Erdélyi, Magnus, Oberhettinger, and Tricomi, *Tables of Integral Transforms* (McGraw-Hill Book Company, Inc., New York, 1954), Volumes I and II.
3. Definitions of the transforms used in the present work are given in Appendix I.
4. D. V. Widder, *Advanced Calculus* (Prentice-Hall, Inc., New York, 1947); Trans. Am. Math. Soc. **36**, 107 (1934). I. I. Hirschman and D. V. Widder, *The Convolution Transform* (Princeton University Press, Princeton, 1955); this book affords a rigorous mathematical background for many of the relations considered in the present work.
5. N. F. Mott and I. N. Sneddon, *Wave Mechanics and Its Applications* (Oxford University Press, New York, 1948), p. 69.
6. W. Schützer and J. Tiomno, Phys. Rev. **83**, 249 (1951).
7. Gell-Mann, Goldberger, and Thirring, Phys. Rev. **95**, 1612 (1954).
8. N. G. Van Kampen, Physica **21**, 127 (1955).
9. R. Simha, J. Appl. Phys. **13**, 201 (1942).
10. B. Gross, J. Appl. Phys. **18**, 212 (1947).
11. B. Gross, J. Appl. Phys. **19**, 257 (1948).
12. B. Gross and H. Pelzer, J. Appl. Phys. **22**, 1035 (1951).
13. B. Gross, *Theory of Viscoelasticity* (Hermann et Cie, Paris, 1953).
14. E. A. Guillemin, *The Mathematics of Circuit Analysis* (John Wiley and Sons, Inc., New York, 1949), see especially pp. 338–343.
15. H. Fröhlich, *Theory of Dielectrics* (Oxford University Press, New York, England, 1949).
16. J. R. Macdonald, J. Chem. Phys. **20**, 1107 (1952).
17. C. J. Gorter and R. Kronig, Physica **3**, 1009 (1936).
18. G. E. Pake and E. M. Purcell, Phys. Rev. **74**, 1184 (1948).
19. R. Lacroix, Helv. Phys. Acta **27**, 283 (1954).
20. R. Kronig, J. Opt. Soc. Am. **12**, 547 (1926).
21. H. A. Kramers, Atti Congr. dei Fisici, Como, 545 (1927).
22. T. Murakami and M. S. Corrington, RCA Rev. **9**, 602 (1948).
23. M. F. Gardner and J. L. Barnes, *Transients in Linear Systems* (John Wiley & Sons, Inc., New York, 1942), pp. 234, 263.
24. Reference 23, pp. 255, 365. The unit impulse is often called the Dirac delta function although it has been used for well over a hundred years.
25. B. Van Der Pol and H. Bremmer, *Operational Calculus Based on the Two-Sided Laplace Integral* (Cambridge University Press, New York, 1955). This reference contains a thorough discussion of the impulse function and its physical approximations.
26. Reference 23, pp. 100, 115.
27. Some useful mathematical relations obeyed by the unit step and unit impulse are summarized in Appendix II.
28. Reference 23, p. 228.
29. J. G. Truxal, *Automatic Feedback Control System Synthesis* (McGraw-Hill Book Company, Inc., New York, 1955), pp. 59–71
30. Reference 23, pp. 131–133.
31. M. K. Brachman and J. R. Macdonald, Physica **20**, 1266 (1954).
32. M. K. Brachman and J. R. Macdonald, Physica **22**, 141 (1956).
33. Immittance is a general term for either impedance or admittance.
34. J. R. Carson, *Electric Circuit Theory and the Operational Calculus* (Chelsea Publishing Company, New York, 1926), second edition, p. 14, et. seq.
35. Reference 15, p. 6.
36. I. Gumowski, Proc. Inst. Radio Engrs. **42**, 819 (1954).
37. K. S. Miller, Trans. Inst. Radio Engrs. Professional Group on Circuit Theory **CT-2**, 26 (1955).
38. Reference 29, p. 57.
39. For completeness, it is also worth noting that the impulse response of a linear system is equal to the cross-correlation between system input and output when the system is excited by white noise. When cross-correlation is carried out electronically, this relationship affords a convenient way of obtaining the impulse response directly, independent of system noise (see reference 29, pp. 437–438).
40. See Appendix II where the connection of this problem with impulse functions is further discussed.
41. The constant σ in $p=\sigma+i\omega$ is usually positive or zero. When convergence is obtained with a negative σ, however, the limiting expressions for $\delta(\omega)$ and $\delta'(\omega)$ require $\lim \sigma \to 0+$; that is, σ must approach zero from the right.
42. Corrington, Murakami, and Sonnenfeldt, RCA Rev. **15**, 389 (1954), have recently given the following results (rewritten in the present notation) related to our Eqs. (23) and (24):

$$\frac{T(\omega)}{\omega} = \int_0^\infty [A(t) - A(\infty)] \cos\omega t dt,$$

$$\frac{P(\omega)}{\omega} = \frac{P(0)}{\omega} + \int_0^\infty [A(t) - A(\infty)] \sin\omega t dt.$$

The treatment of these authors made use of the Fourier integral and so led to results without the convergence factors of the present work. If these factors are reinstated, the integration over the $A(\infty)$ terms may be carried out.

On making use of the result $A(\infty)=P(0)$ (see Appendix III) and using (23) and (24), we find that the above equations become

$$\frac{T(\omega)}{\omega} = J(\omega) - \pi P(0)\delta(\omega),$$

$$\frac{P(\omega)}{\omega} = H(\omega).$$

The first equation is identical with our (17″) and is hence valid in most cases of interest. The second equation differs significantly from (18″), however, and is incorrect, for ex-

ample, when $T(\omega)=-1/\omega$. The omission of the $\delta(\omega)$ term in the expression for $P(\omega)/\omega$ obtained by the above authors arises from their setting such expressions as $\lim_{\sigma\to 0}\int_0^\infty e^{-\sigma t}\cos\omega t dt$ to zero instead of to a delta function. Such delta-function omissions cause many of their results to lack the $J(\infty)$ and $P(\infty)$ terms appearing in the formulas of the present work.
43. The fact that ideal reactive elements yield immittances with nonzero parts was first pointed out by B. Gross, Symposium on New Research Techniques in Physics, Rio de Janeiro, July 15–29, 1952, and has also been discussed by E. A. Guillemin, Proc. Natl. Electronics Conf. **9**, 513 (1953).
44. D. Bierens de Haan, *Nouvelles Tables d'Integrales Définies* (G. E. Stechert & Company, New York, 1939), corrected edition of 1867.
45. E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford University Press, New York, 1937), second edition, p. 120.
46. The Kronig-Kramers relations are sometimes written in the following form which eliminates the pole at $y=\omega$ and allows the principal-value signs to be omitted.

$$J(\omega)=J(\infty)+\frac{2}{\pi}\int_0^\infty\frac{[yH(y)-\omega H(\omega)]}{y^2-\omega^2}dy,$$

$$H(\omega)=\frac{2\omega}{\pi}\int_0^\infty\frac{[J(y)-J(\omega)]}{\omega^2-y^2}dy.$$

The usual justification for this procedure is that the integral $(2/\pi)\int_0^\infty dy/(y^2-\omega^2)$ is zero, and this result is even given in some tables of integrals without qualification.[44] Actually the integral equals $\pi\delta(\omega)$ and is only zero for $\omega\neq 0$. The reason the above procedure is normally valid is that one is actually adding a term such as $\omega\delta(\omega)J(\omega)$ or $\omega\delta(\omega)H(\omega)$. Since $\omega^n\delta(\omega)$ $(n=1,2,3\cdots)$ is identically zero, the added terms are also usually zero. This procedure must be applied with circumspection, however, since for either $S(p)=p^{-1}$ or 1, $T(\omega)$, $J(\omega)$, and/or $H(\omega)$ will contain terms involving ω^{-1} or ω^{-2}, and products such as $\omega\delta(\omega)H(\omega)$ will no longer necessarily be identically zero.
47. See Appendix I.
48. H. W. Bode, *Network Analysis and Feedback Amplifier Design* (D. Van Nostrand Company, Inc., New York, 1945), pp. 303–336.
49. B. Gross, Am. Math. Monthly **50**, 90 (1943). This paper was written before it became apparent that $S(i\omega)$ might sometimes involve impulse functions (see reference 43); thus, it is not applicable to the entire class of Kronig-Kramers transformable functions.
50. Reference 48, p. 335.
51. W. Cauer, Bull. Am. Math. Soc. **38**, 713 (1932).
52. P. I. Richards, Duke Math. J. **14**, 777 (1947).
53. L. Schwartz, *Théorie des Distributions* (Hermann et Cie, Paris, 1950, 1951), volumes I and II.
54. W. Güttinger, Phys. Rev. **89**, 1004 (1953); Z. Naturforsch. **10a**, 257 (1955).
55. J. J. Smith and P. L. Alger, Trans. Am. Inst. Elec. Engrs. part II, **68**, 939 (1949).
56. W. Güttinger, Progr. Theoret. Phys. **13**, 612 (1955).
57. A partial exception to this rule is the solid line (15′) leading from $H(\omega)$ to $P(\omega)$. The relation (15′) has not been rigorously justified by our analysis. Further, this path only leads to $P(\omega)$. There are no solid paths originating at $P(\omega)$ which could allow complete information about the system to be obtained from $H(\omega)$ or $P(\omega)$ alone. Guillemin[43] has stated that $P(\omega)$ is sufficient to completely characterize the system. The present analysis shows this to be correct only when $J(\infty)=0$, not in general. It is also worth pointing out that Gross[43] inclines to the use of $H(\omega)$ to characterize the system even though $J(\infty)$ must be known in addition because, as he has shown, $H(\omega)$ may be interpreted in a natural way as the distribution function of canonical reactance systems while $J(\omega)$ apparently cannot be so interpreted.
58. C. J. F. Böttcher, *Theory of Electric Polarisation* (Elsevier Publishing Company, Amsterdam, 1952).
59. C. J. Gorter, *Paramagnetic Relaxation* (Elsevier Publishing Company, Amsterdam, 1947).
60. Reference 48, pp. 103–136.
61. O. Brune, J. Math. and Phys. **10**, 191 (1931).
62. Reference 48, p. 121.
63. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), pp. 353, 357.
64. Reference 14, p. 339.
65. Reference 63, p. 372.
66. Reference 63, p. 373.
67. Reference 15, pp. 90–92.
68. W. Kauzmann, Revs. Modern Phys. **14**, 12 (1942).
69. R. M. Fuoss and J. G. Kirkwood, J. Am. Chem. Soc. **63**, 385 (1941).
70. Reference 45, pp. 317–319.
71. Reference 2, Volume II, p. 213.
72. Reference 45, p. 316, Eq. (11.7.3).
73. Reference 45, p. 318, Eq. (11.8.4).
74. Reference 45, p. 315.
75. I. N. Sneddon, *Fourier Transforms* (McGraw-Hill Book Company, Inc., New York, 1941), p. 43, Eq. (102).
76. R. H. Cole and K. S. Cole, J. Chem. Phys. **9**, 341 (1941).
77. D. W. Davidson and R. H. Cole, J. Chem. Phys. **20**, 1389 (1952).
78. R. L. Pritchard, Proc. Inst. Radio Engrs. **42**, 786 (1954).
79. Reference 23, p. 241.
80. See Appendix II.
81. B. Gross and H. Pelzer, Proc. Roy. Soc. (London) **A210**, 434 (1951).
82. Reference 15, p. 101.
83. J. H. Van Vleck and V. F. Weisskopf, Revs. Modern Phys. **17**, 227 (1945).
84. M. A. Garstens, Phys. Rev. **93**, 1228 (1954).
85. Garstens, Singer, and Ryan, Phys. Rev. **96**, 53 (1954).
86. R. K. Wangsness, Phys. Rev. **98**, 927 (1955).
87. H. A. Lorentz, *Theory of Electrons* (Teubner, Leipzig, 1906).
88. M. Lerch, Acta Mathamatica **27**, 339 (1903).
89. Reference 1, p. 26.
90. E. V. Schweidler, Ann. Physik **24**, 711 (1907).
91. B. Gross, Z. Physik **9–10**, 598 (1938).
92. K. S. Cole and R. H. Cole, J. Chem. Phys. **10**, 98 (1942).
93. Davidson, Auty, and Cole, Rev. Sci. Instr. **22**, 678 (1951).
94. A. van der Ziel, *Noise* (Prentice-Hall, Inc., New York, 1954).
95. Goldberger, Miyazawa, and Oehme, Phys. Rev. **99**, 986 (1955).
96. H. L. Anderson, Revs. Modern Phys. **27**, 269 (1955).
97. H. T. Davis, *The Theory of Linear Operators* (Principia Press, Bloomington, 1936).
98. Reference 29, pp. 29–34.
99. Reference 14, pp. 263–264, 351–360.
100. W. R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill Book Company, Inc., New York, 1939).
101. Note that the result for $x^{-1}\circ\delta^{(m)}$ given as Eq. (2.25) in reference 56 is incorrect in general, as may be shown by direct expansion using Güttinger's Eq. (2.11), which is basic. A closed result for $x^{-1}\circ\delta^{(m)}$ for all m has not yet been found.