change in n, for example, and not in any electrical or emissive properties. Coarse and fine particles from the same lot do vary somewhat in activation as shown by the fact that their emission color often changes systematically with the mean particle size of the fraction separated.[20]

The calculations in this paper indicate, however, that it may be sufficient to consider only effects due to phosphor particle dispersion in the dielectric material to explain variations in brightness, the parameters L_0 and V_0, and efficiency, with particle size.

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Theory and Application of a Superposition Model of Internal Friction and Creep*

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Energy dissipation in solids is important in both transient and steady-state measurements. The results of such measurements can be associated with a distribution of relaxation times provided the material is linear. In the present work, general relations are derived for the attenuation factor, phase factor, and specific dissipation function $1/Q$ pertaining to transmission of small amplitude stress waves in a material describable by a distribution of relaxation times. Next, a specific, physically realizable relaxation time distribution is used to obtain a creep function and to relate transient creep and frequency response measurements. Curves of $1/Q$ vs frequency are calculated with a digital computer and show a region approximately proportional to frequency at sufficiently low relative frequencies, a region of virtual frequency independence, and a final region proportional to inverse frequency at high relative frequency. The relation of the present work to other treatments of creep and internal friction is discussed, and the applicability is examined of the analytic and numeric results to creep measurements on metals and rocks, to low-frequency wave transmission in the earth, to other damping results for the earth as a whole, and to higher-frequency wave transmission and vibration results for geophysical and other solids. Good agreement between theory and experiment is found for frequency regions where adequate data are available, indicating that all the damping phenomena considered may be well described by a linear theory in the range of very small strain.

INTRODUCTION

When a solid is set in motion, some of the elastic energy of the material is dissipated as heat. The various means by which such energy loss occurs are collectively known as internal friction.[1] The presence and magnitude of internal friction can be inferred from the results of two different but related kinds of experiments which involve, respectively, the transient and the steady-state response of the material. When a very small constant mechanical stress is suddenly applied to a solid, there results an "instantaneous" strain followed by a retarded deformation whose rate continuously decreases. The "instantaneous" strain, which is actually transmitted with the speed of sound in the medium, is the usual Hooke's law linear response. The remaining response, which is termed transient creep and which occurs particularly at temperatures low compared to the melting point and at low strain amplitudes, is characteristic of viscoelastic behavior. Internal friction can also often be determined from a transient experiment in which the sample is initially set vibrating and the logarithmic decrement of its amplitude decay is determined. This method is not very convenient when a wide range of frequencies must be covered. Alternatively, energy loss occurs when a steady-state stress wave is transmitted through the sample, and the magnitude of such loss can be related to the attenuation factor of the waves.[1]

The transient and steady-state responses of a system are intimately associated when the system is linear and either can be calculated when the other is known.[2] It is often useful, however, to make measurements of both kinds on the same material since they usually cover different time spans. For the study of materials exhibiting short relaxation times, steady-state measurements with periods less than one second are appropriate.

* A part of the present work, consisting mainly of the figures and some of the mathematical results but none of the present analysis, discussion, and interpretation was presented jointly by C. Lomnitz and the author at the 1960 Helsinki meeting of the International Geophysical Union.

[1] H. Kolsky, *Stress Waves in Solids* (Clarendon Press, Oxford, England, 1953). This reference gives a good introduction to internal friction and the various ways it can be measured. A summary of modern methods of measuring dynamic elastic properties is also given by K. W. Hillier, *Stress Wave Propagation in Materials*, edited by N. Davids (Interscience Publishers, Inc., New York, 1960), p. 183.

[2] J. R. Macdonald and M. K. Brachman, Revs. Modern Phys. 28, 393 (1956).

On the other hand, creep measurements can extend for a week or greater and are useful when a material with long relaxation times must be investigated.

In the present work, we shall consider the interrelation of transient creep and steady-state measurements for a viscoelastic solid exhibiting a distribution of relaxation times. This distribution will be selected, in so far as possible, to make the resulting theoretical creep and wave-attenuation functions agree with experimental results. The entire treatment is based on the validity of the principle of superposition and hence on the linear behavior of the material. The theory is therefore applicable only for small strains and small wave amplitudes. Brinkman and Schwarzl[3] and Knopoff and MacDonald[4] have given nonlinear theories of energy loss in solids. Although all materials are of course nonlinear, it is reasonable to expect that at sufficiently small strains nonlinear behavior can be neglected and that only linear response need be considered. It will be shown in this work that energy dissipation in such a limitingly linear system can explain considerable experimental creep and wave attenuation results, and that therefore nonlinear effects need not be invoked to describe such results.

In the next section, some possible creep functions which imply a distribution of relaxation times will be considered. Then, in the following section, general relations between transient and steady-state response functions will be presented and a specific, physically realizable creep function obtained. Finally, quantitative predictions of the theoretical work obtained from a digital computer calculation will be discussed and compared with various experimental data on solids.

THE CREEP FUNCTION

The response of a linear mechanical system to a suddenly applied stress was formulated as an empirical law by Boltzmann.[5] The result is an instance of the real convolution[6] or superposition integral well known[7] to mathematicians before its use by Boltzmann. For a linear viscoelastic material subjected to a stress $\sigma(t)$ suddenly applied at $t=0$, the resulting strain is

$$\epsilon(t)=M^{-1}\left[\sigma(t)+\int_{0-}^{t}\sigma(\tau)A(t-\tau)d\tau\right], \qquad (1)$$

where M is the appropriate elastic modulus, and the lower limit is here given as $0-$ to include the effects of any impulse functions centered at $t=0$ contained in $\sigma(t)$. This limit can be extended to $-\infty$ if desired.[2] The quantity $A(t)$ is a memory or rate-of-creep function and its appearance in the part of the response associated with creep shows that the over-all behavior of the material at a given time greater than zero is influenced by past states of the system. The strain resulting from a constant stress applied at $t=0$, $\sigma(t)=\sigma_0 u(t)$, is thus

$$\epsilon(t)=\sigma_0 M^{-1}[u(t)+\psi(t)], \qquad (2)$$

where

$$\psi(t)\equiv\int_0^t A(x)dx \qquad (3)$$

and $u(t)$ is the unit step function defined so that $u(0-)=0$ and $u(t>0)=1$. The quantity $\psi(t)$ is usually called the creep function.

A number of authors[2,8–16] have treated the relationships between such defining functions as $\psi(t)$ or $A(t)$ and other response or describing functions of the pertinent linear system. Such relations apply formally to the description of either viscoelastic or dielectric behavior. For the viscoelastic body, the useful application of theoretical results to the analysis of experimental data depends upon a felicitous choice of $\psi(t)$. Many explicit functions have been proposed, with forms of the power law[17–19] and logarithmic law[5,19–22] being found particularly useful. Some time ago, creep applications of the modified logarithmic law

$$\psi(t)=q\ln[1+(t/\tau_0)], \qquad (4)$$

where q and τ_0 are constant, were discussed by Lyons.[22] More recently, Lomnitz[23] independently suggested this expression and showed that it could be used to describe his experimental results for creep in igneous rocks under shear stress. Unfortunately, measurements were not extended to times such that (t/τ_0) was of the order of or less than unity; thus, a complete comparison between (4) and experiment was impossible and no distinction could be made between (4) and simple *lnt* dependence. The latter dependence clearly cannot hold in the limit

[3] H. C. Brinkman and F. Schwarzl, Discussions Faraday Soc. 23, 11 (1957).

[4] L. Knopoff and G. J. F. MacDonald, J. Geophys. Research 65, 2191 (1960).

[5] L. Boltzmann, Pogg. Ann. Phys. Lpz. 7, 624 (1876); Sitzber K. Akad. Wiss. Wien, Math.-Naturwiss. Classe 70, 275 (1874).

[6] I. I. Hirschmann and D. V. Widder, *The Convolution Transform* (Princeton University Press, Princeton, New Jersey, 1955).

[7] M. F. Gardner and J. L. Barnes, *Transients in Linear Systems* (John Wiley & Sons, Inc., New York, 1942), pp. 364–365.

[8] V. Volterra, Ann. Ecole norm. super. 24, 401 (1907).

[9] B. Derjaguine, Beitr. angew. Geophys. 4, 452 (1934).

[10] C. Zener, *Elasticity and Anelasticity of Metals* (University of Chicago Press, Chicago, Illinois, 1948).

[11] B. Gross, *Mathematical Structure of the Theories of Viscoelasticity* (Hermann & Cie, Paris, France, 1953).

[12] E. R. Love, Australian J. Phys. 9, 1 (1956).

[13] J. Schrama, "On the phenomenological theory of linear relaxation processes," dissertation (Leiden, 1957).

[14] H. Konig and J. Meixner, Math. Nachr. 19, 265 (1958).

[15] D. R. Bland, *The Theory of Linear Viscoelasticity* (Pergamon Press, New York, 1960).

[16] S. C. Hunter, *Progress in Solid Mechanics* edited by I. N. Sneddon and R. Hill (Interscience Publishers, Inc., New York, 1960), Vol. I, p. 3.

[17] E. N. daC. Andrade, Proc. Roy. Soc. (London) A84, 1 (1910).

[18] B. Gross, J. Appl. Phys. 18, 212 (1947).

[19] B. J. Rigby, Brit. J. Appl. Phys. 11, 281 (1960).

[20] F. Phillips, Phil. Mag. 9, 513 (1904).

[21] D. T. Griggs, J. Geol. 47, 225 (1939).

[22] W. J. Lyons, J. Appl. Phys. 17, 472 (1946).

[23] C. Lomnitz, J. Geol. 64, 473 (1956).

of short times. Later, Lomnitz[24] showed theoretically that (4) led to approximate frequency independence of the specific dissipation, or internal friction factor $1/Q$, in good agreement with many observations on the attenuation of stress waves in solids. Earlier, Bennewitz[25] carried out calculations of the steady-state response with a creep function of the form (4).

Jeffreys[26,27] has recently proposed a simple generalization of (4),

$$\psi(t)=q\nu^{-1}[\{1+(t/\tau_0)\}^{\nu}-1], \tag{5}$$

where ν is a constant which will here be restricted to the range $0\leq\nu<1$. Some physical interpretation of the parameters of (5) will be given later. Equation (5) reduces to the logarithmic form (4) in the limit $\nu\rightarrow 0$. A creep function of the form of (5) but with $\nu\leq 0$ has also been used to describe the behavior of polymeric materials and has been derived theoretically from consideration of the microscopic processes which may occur during elongation under stress.[19,28] The $A(t)$ function corresponding to (5) is

$$A(t)=(q/\tau_0)[1+(t/\tau_0)]^{\nu-1}. \tag{6}$$

An expression of this form was first proposed by Voglis[29] to describe the dielectric analogue of viscoelastic creep—the time dependence of the charging and discharge currents of dielectrics. Further dielectric applications of equations related to (6), including the case $\nu<0$, will be discussed elsewhere by the author.

The form (5) is a generalization of both the simple power law and the logarithmic form and has the advantage over the former that it leads to a finite rather than infinite initial rate of creep. Unfortunately, both (4) and (5) imply infinite final strain for constant applied stress. Ways of modifying (5) to achieve a form which predicts a finite final strain and thus allows it to describe a physically realizable linear solid have been discussed by the author.[30] The consequences of one such modification of (5) will be considered in some detail in the present work. The expression (5) is picked as a starting point both because of its considerable generality and because, as will be shown, its physically realizable modification can explain an appreciable amount of transient and steady-state data.

[24] C. Lomnitz, J. Appl. Phys. 28, 201 (1957). The square-root signs in Eqs. (27), (28), (33), and (36) are incorrect and should be eliminated. Their elimination makes only a small difference in the $1/Qq$ curves shown, however. This error is corrected in the present work.

[25] K. Bennewitz, Phys. Z. 25, 417 (1924).

[26] H. Jeffreys, Geophys. J. 1, 92 (1958).

[27] H. Jeffreys, Monthly Notices Roy. Astron. Soc. 118, 14 (1958).

[28] H. Burte and G. Halsey, Textile Research J. 17, 465 (1947).

[29] G. M. Voglis, Z. Phys. 109, 52 (1938).

[30] J. R. Macdonald, J. Appl. Phys. 30, 453 (1959). In this reference the present $\psi(t)$ was denoted $\phi(t)$. The notation has been changed in the present work to agree with that of the majority of recent authors on viscoelastic relations.

SYSTEM ANALYSIS

In this section we shall present the equations which connect a given creep function with other functions and observables of the described system. In particular, general relationships will be derived between the phase and attenuation factors of a stress wave in a solid and the sinusoidal energy storage and loss factors of the material. Then, Eq. (5) will be modified to make it apply to a physically realizable linear system, and pertinent measurable quantities will be derived and evaluated.

Equation (1) shows that $A(t)$ is the impulse response of $\epsilon(t)$ for the creeping part of the system, since if $\sigma(t)$ is taken as $\sigma_1\delta(t)$, where $\delta(t)$ is the Dirac delta function, (1) leads to

$$\epsilon(t)=(\sigma_1/M)[\delta(t)+A(t)]. \tag{7}$$

On differentiating (1), one obtains

$$\frac{d\epsilon}{dt}=\frac{1}{M}\left[\frac{d\sigma}{dt}+\sigma(0-)A(t)+\int_{0-}^{t}\frac{d\sigma(\tau)}{d\tau}A(t-\tau)d\tau\right]. \tag{8}$$

This equation indicates that $A(t)$ is also the creep contribution to the step-function response of $d\epsilon/dt$; for if $\sigma(t)=\sigma_0 u_0(t)$, then

$$d\epsilon/dt=(\sigma_0/M)[\delta(t)+A(t)]. \tag{9}$$

In both (7) and (9), the $\delta(t)$ term represents the elastic or Hookean part of the response and the $A(t)$ term the viscoelastic part which is associated with creep.

If now a sinusoidally varying stress is applied to the system, the work per unit volume done in a cycle of period t_0 is

$$\Delta W=\int_0^{t_0}\sigma(t)\frac{d\epsilon(t)}{dt}dt. \tag{10}$$

Equations (8) and (10) now allow us to identify $A(t)=d\psi/dt$ with the indicial admittance[2] of the system, and we shall use the $A(t)$ and $\psi(t)$ notation and the notation of reference 2 hereafter.

The Laplace transform of $A(t)$ defines the network function

$$Q(p)\equiv\mathcal{L}[A(t)]=\int_{0-}^{\infty}A(t)e^{-pt}dt, \tag{11}$$

where p is a complex frequency variable. For $A(t)$ functions without singularities at the origin, $Q(0)=\psi(\infty)$, where $(\sigma_0/M)\psi(\infty)$ is the final strain arising from creep when a step function of stress is applied at $t=0$. For sinusoidal excitation of the form $\sigma(t)=\sigma_0 e^{i\omega t}$, the real part of the complex frequency variable p may be taken arbitrarily small and the network function $Q(p)$ (not to be confused with the quality factor appearing in the specific dissipation $1/Q$) may be separated into real and imaginary parts,

$$Q(p)\rightarrow Q(i\omega)\equiv J(\omega)-iH(\omega). \tag{12}$$

The dynamic or complex compliance relating $\epsilon(i\omega t)$ and $\sigma(i\omega t)$ is then $M^{-1}[1+Q(i\omega)]$.

For sinusoidal applied stress, we may follow the procedure of Collins and Lee[31] to obtain the following one-dimensional equation for displacement $U(x,t)$ for plane waves incident on an isotropic material describable by superposition,

$$V_e^2\frac{\partial^2 U}{\partial x^2}=\frac{\partial^2 U}{\partial t^2}+\frac{\partial^2 U}{\partial t^2}\star A, \tag{13}$$

where the star denotes the convolution transform,[2,6,7] the relations $\epsilon=\partial U/\partial x$, Eq. (1), and $\partial\sigma/\partial x=\rho\partial^2 U/\partial t^2$ have been used; ρ is the material density, and $V_e=(M/\rho)^{\frac{1}{2}}$ is the elastic (zero dissipation) phase velocity.

On taking the Laplace transform of (13) with respect to t, one finds

$$\frac{d^2u}{dx^2}=\frac{p^2[1+Q(p)]u}{V_e^2}\equiv\gamma^2(p)u, \tag{14}$$

where $u(x)=\mathfrak{L}[U(x,t)]$. For given initial conditions and a physically realizable form of $Q(p)$, Eq. (13) can, in principle, be solved for $U(x,t)$. Here, we are primarily interested in the x dependence of the plane wave in the material. From (14), this dependence for decaying waves will be of the form

$$u(x)=u_0e^{-\gamma x}\equiv u_0e^{-ikx}\equiv u_0e^{-(\alpha+i\beta)x}, \tag{15}$$

where k is the wave number and $\gamma(i\omega)=\alpha+i\beta$ is the transmission factor. Here α is the attenuation factor, and the phase factor is $\beta=\omega/V$, where V is the phase velocity.

Equations (12) and (14) lead to

$$\alpha(\omega)+i\beta(\omega)=(i\omega/V_e)[1+J(\omega)-iH(\omega)]^{\frac{1}{2}}. \tag{16}$$

On separating real and imaginary parts, one finds the general relations[32]

$$\alpha^2=\frac{1}{2}\left(\frac{\omega}{V_e}\right)^2[1+J(\omega)]\left\{\left[1+\left(\frac{H(\omega)}{1+J(\omega)}\right)^2\right]^{\frac{1}{2}}-1\right\}, \tag{17}$$

$$\beta^2=\frac{1}{2}\left(\frac{\omega}{V_e}\right)^2[1+J(\omega)]\left\{\left[1+\left(\frac{H(\omega)}{1+J(\omega)}\right)^2\right]^{\frac{1}{2}}+1\right\}. \tag{18}$$

In the zero dissipation case, $J(\omega)$ and $H(\omega)$ are zero, and (18) yields $\beta=\omega/V_e$, the correct result for this case. Note that Eq. (18), together with the definition of β, allows one to calculate the phase velocity V when creep and loss are present and described by the functions $J(\omega)$ and $H(\omega)$.

Either by analogy with the dielectric case or by using Eqs. (8) and (10) directly, one can show[33] that the energy dissipation per cycle in the system for sinusoidal driving stress is proportional to $H(\omega)$ and the maximum stored energy is similarly proportional to $[1+J(\omega)]$. Since the specific dissipation factor is the ratio of energy dissipated in a cycle to the maximum energy stored during a cycle, it may be written as

$$1/Q\equiv H(\omega)/[1+J(\omega)], \tag{19}$$

allowing (17) and (18) to be rewritten in terms of $J(\omega)$ and $(1/Q)^2$. Then these equations may be combined to yield the exact result

$$\frac{1}{Q}=\frac{2\alpha/\beta}{1-(\alpha/\beta)^2}, \tag{20}$$

which reduces to the usual approximate form, $1/Q\cong 2\alpha/\beta$, in the small-dissipation case. Note that $1/Q$ is independent of stress amplitude as expected for a linear system.

When the functional form of $\psi(t)$ or $A(t)$ is known, one can obtain $J(\omega)$ and $H(\omega)$ by carrying out Fourier cosine and sine transforms[2] of $A(t)$. In order, however, to obtain equations which are most convenient for computation and which represent a physically realizable system, we shall work with a distribution of relaxation times function $G_1(\tau)$. Actually, this function here describes a distribution of retardation times,[11,13,32] but this distinction, a matter of nomenclature, will not be stressed herein. In the present case, it will prove convenient to use the variable $z\equiv\tau_0/\tau$, where τ_0 is a fixed relaxation time such as that occurring in Eq. (5). When $G(z)\equiv G_1(\tau)$ is known, the following quantities may be calculted from it[2]:

$$Q(p)=\tau_0\int_0^\infty\frac{z^{-1}G(z)dz}{z+(p\tau_0)}, \tag{21}$$

$$J(W)=\tau_0\int_0^\infty\frac{G(z)dz}{z^2+W^2}, \tag{22}$$

$$H(W)=W\tau_0\int_0^\infty\frac{z^{-1}G(z)dz}{z^2+W^2}, \tag{23}$$

$$A(T)=\int_0^\infty z^{-1}G(z)e^{-Tz}dz, \tag{24}$$

$$\begin{aligned}\psi(T)&=\tau_0\int_0^T A(x)dx\\&=\tau_0\int_0^T\int_0^\infty z^{-1}G(z)e^{-xz}dzdx\\&=\tau_0\int_0^\infty z^{-2}G(z)\{1-e^{-Tz}\}dz,\end{aligned} \tag{25}$$

[31] F. Collins and C. C. Lee, Geophysics 21, 16 (1956).

[32] Some time after the derivation of (17) and (18) in 1958 it was found that equivalent expressions had been published by T. Alfrey, Jr. and E. F. Gurnee, *Rheology*, edited by F. R. Eirich (Academic Press, Inc., New York, 1956), Vol. 1, p. 387.

[33] B. Gross, J. Appl. Phys. 19, 257 (1947).

on interchanging the order of integration in (25) and integrating with respect to x. In these equations, $W\equiv\omega\tau_0$ and $T\equiv t/\tau_0$. To avoid introducing new symbols, we have not distinguished between such functions as $J(\omega)\equiv J(W/\tau_0)$ and $J(W)$ in Eqs. (22)–(25) and throughout the remainder of the paper. If $A(T)$ is known, $G(z)$ itself can be readily obtained on inversion of (24), yielding

$$z^{-1}G(z)=\mathfrak{L}^{-1}[A(T)], \tag{26}$$

where $\mathfrak{L}^{-1}$ is the inverse Laplace transform operator, here involving the variable T instead of the usual p.

The $A(T)$ function corresponding to Eq. 6 is

$$A(T)=q\tau_0^{-1}[1+T]^{\nu-1}. \tag{27}$$

The associated $G(z)$ is, from (26),

$$G(z)=(q/\tau_0)z^{1-\nu}e^{-z}/\Gamma(1-\nu). \tag{28}$$

An equivalent form was first given by Voglis.[29] This result, in turn, leads to

$$Q(p)=q(p\tau_0)^{-\nu}e^{p\tau_0}\Gamma(\nu,p\tau_0), \quad (\nu<1), \tag{29}$$

which involves the incomplete gamma function,[34] a fact first mentioned by Jeffreys.[27] Equations (22) and (23) yield

$$J(W)=\frac{q}{\Gamma(1-\nu)}\int_0^\infty\frac{z^{1-\nu}e^{-z}dz}{z^2+W^2}, \tag{30}$$

$$H(W)=\frac{qW}{\Gamma(1-\nu)}\int_0^\infty\frac{z^{-\nu}e^{-z}dz}{z^2+W^2}. \tag{31}$$

The last two integrals may be expressed in terms of Lommel functions of a single variable,[35]

$$J(W)=q|W|^{\frac{1}{2}-\nu}(1-\nu)S_{\nu-\frac{3}{2},\frac{1}{2}}(|W|), \tag{30'}$$

$$H(W)=q|W|^{\frac{1}{2}-\nu}(\mathrm{sgn}W)S_{\nu-\frac{1}{2},\frac{1}{2}}(|W|), \tag{31'}$$

but these functions are not sufficiently well tabulated to be useful over the entire ranges of W and ν of interest. For large W, they lead to $J(W)\rightarrow q(1-\nu)/W^2$ and $H(W)\rightarrow q/W$.

A more important difficulty is that the $G_1(\tau)$ corresponding to (28) cannot be normalized when $\nu\geq 0$. It is easy to prove that

$$J(0)=Q(0)=\psi(\infty)=\int_0^\infty G_1(\tau)d\tau, \tag{32}$$

but (30) shows that $J(0)=\infty$, consistent with $\psi(\infty)=\infty$ from (5). This failure of normalization means that (27) and (28) do not describe a physically realizable relaxation system. In fact, $\psi(\infty)=\infty$ implies an infinite number or concentration of relaxation times, an impossible requirement for a finite piece of matter.

In previous work,[30] various modifications of $G_1(\tau)$ or $A(t)$ were suggested which would lead to a physically realizable system without appreciable alteration of $\psi(t)$ in the main region of times usually accessible to observation. The lack of convergence of (32) in the present case arises from too slow a decay of the density of relaxation times at long τ's or short z values. As discussed previously, normalization can be achieved if an increase in the decay rate of $G_1(\tau)$ is made for long τ's exceeding a specific value which may be called τ_∞. Such an increase will result in a smaller density of relaxation times for $\tau>\tau_\infty$ than predicted by (28) and can lead to finite final strain. Dealing again with the z variable, convergence is assured if we take, for example,

$$\left.\begin{aligned}G(z)&=(q/\tau_0)a^{1-\nu}e^{-a}(z/a)^2/\Gamma(1-\nu), \quad 0\leqslant z\leqslant a,\\ G(z)&=(q/\tau_0)z^{1-\nu}e^{-z}/\Gamma(1-\nu), \quad z\geqslant a,\end{aligned}\right\} \tag{33}$$

where $a\equiv\tau_0/\tau_\infty$ and will usually be very small compared to unity. For $z\leq a$, the initial slope of $G(z)$ on a log-log plot is two instead of the value $(1-\nu)$ which follows from (28). Any value greater than one could have been used to ensure convergence of (32). Were experimental creep data available for such long times that the actual distribution of very long relaxation times could be inferred, a different slope than the value here used or a different form of the distribution function in this region might prove preferable. One possibility would be to set $G(z)=0$ for $z\leq a$. In the absence of such data, the present initial slope of two is a reasonable choice. Other alternatives will be discussed in a later paper dealing with dielectric phenomena.

On substituting Eq. (33) in (24), one can obtain the modified rate-of-creep function,

$$A(T)=\frac{q}{\tau_0\Gamma(1-\nu)}\left[\frac{\Gamma(1-\nu,\,a+aT)}{(1+T)^{1-\nu}} + a^{1-\nu}e^{-a}(aT)^{-2}\{1-(1+aT)e^{-aT}\}\right]. \tag{34}$$

When $a\ll 1$, (34) reduces to (27) to good approximation as long as $aT\ll 1$.

It is easiest to obtain $\psi(T)$ from (33) and (25), yielding

$$\psi(T)=\frac{q}{\Gamma(1-\nu)}\left[\Gamma(-\nu,\,a)-\frac{\Gamma(-\nu,\,a+aT)}{(1+T)^{-\nu}} - a^{-\nu}e^{-a}\{-1+(aT)^{-1}(1-e^{-aT})\}\right]. \tag{35}$$

From this expression it follows that $\psi(0)=0$ as it should and that

$$\psi(\infty)=\frac{q}{\Gamma(1-\nu)}[\Gamma(-\nu,\,a)+a^{-\nu}e^{-a}]. \tag{36}$$

[34] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2, pp. 133–143.

[35] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, New York, 1944), 2nd edition, pp. 345–352.

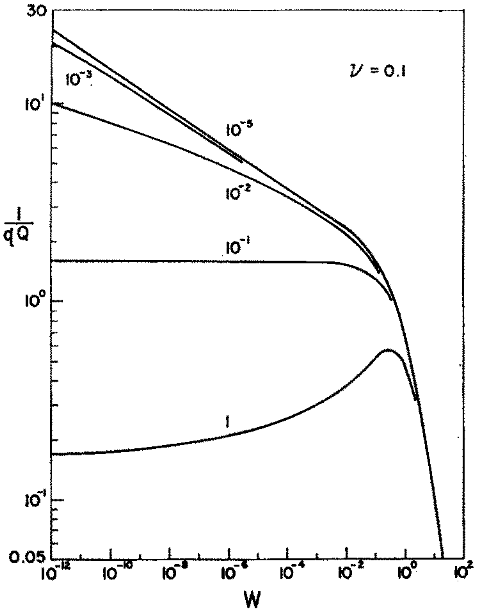

FIG. 1. $1/qQ$ vs normalized frequency W for $\nu=0.1$, $a=0$, and various q values.

Thus, the final strain is finite and positive for $a>0$ as it must be for final mechanical stability. For $a\ll1$, series expansion of the incomplete gamma function leads to

$$\psi(\infty)\cong\begin{cases} q(1-\gamma-\ln a), & \nu=0,\\ \dfrac{q}{\Gamma(1-\nu)}\left[a^{-\nu}\left(1+\nu^{-1}+\dfrac{a}{1-\nu}\right)-\dfrac{\Gamma(1-\nu)}{\nu}\right], & 0<\nu<1, \end{cases} \quad (37)$$

where γ is Euler's constant. In the earlier note[30] some of the terms appearing in (37) were inadvertently omitted.

The $J(W)$ and $H(W)$ functions corresponding to (34) are

$$J(W)=\frac{q}{\Gamma(1-\nu)}\left[a^{-\nu}e^{-a}\left\{1-\left|\frac{W}{a}\right|\tan^{-1}\left|\frac{a}{W}\right|\right\}+\int_a^\infty\frac{z^{1-\nu}e^{-z}dz}{z^2+W^2}\right], \quad (38)$$

$$H(W)=\frac{q}{\Gamma(1-\nu)}\left[a^{-\nu}e^{-a}(W/a)\ln\{1+(a/W)^2\}^{\frac{1}{2}}+W\int_a^\infty\frac{z^{-\nu}e^{-z}dz}{z^2+W^2}\right]. \quad (39)$$

These equations are in a form appropriate for computation with a digital computer and may be used in (19) to yield the dependence of $1/Q$ on W.

When $a\to0$, Eqs. (38) and (39) reduce to (30) and (31). In addition, the main contribution to $J(W)$ and $H(W)$ arises from the integral terms when $100a<W$; the other terms are important when W is of the same order of magnitude or smaller than a. One principal region of interest of W will be $W\ll1$. For $a\ll W\ll1$, the main contributions to the integrals in (38) and (39) will occur for z near W, and we can thus set e^{-z} to unity and a to zero in this case to good approximation. The resulting integrals may be treated as Mellin transforms, leading to

$$\left.\begin{aligned} J(W)&\cong q\Gamma(\nu)\cos(\pi\nu/2)W^{-\nu}, && 0<\nu<2,\\ H(W)&\cong q\Gamma(\nu)\sin(\pi\nu/2)W^{-\nu}, && -1<\nu<1. \end{aligned}\right\} \quad (40)$$

Two cases must now be introduced depending on the magnitude of $J(W)$, which of course depends on $\psi(\infty)$ through (36) as well as on W and ν. Equations (19) and (40) lead to

$$\left.\begin{aligned} &1/Q\cong H(W)\cong q\Gamma(\nu)\sin(\pi\nu/2)W^{-\nu}, && (J(W)\ll1),\\ & && 0<\nu<1,\\ &1/Q\cong H(W)/J(W)\cong\tan(\pi\nu/2), && (J(W)\gg1). \end{aligned}\right\} \quad (41)$$

Note that a $1/Q$ much less than unity, necessary for agreement with experiment, is possible with $J(W)\ll1$ for any value of ν in the range $0<\nu<1$ but that $1/Q\ll1$ requires $\nu\ll1$ when $J(W)\gg1$. Whenever $\nu\ll1$, (41) shows that $1/Q$ will be either completely frequency independent or virtually so.

In the singular case $\nu=0$, one can obtain approximate values of $J(W)$ and $H(W)$ when $a\ll W$ by setting $a=0$ in (38) and (39) and treating the integrals as Laplace

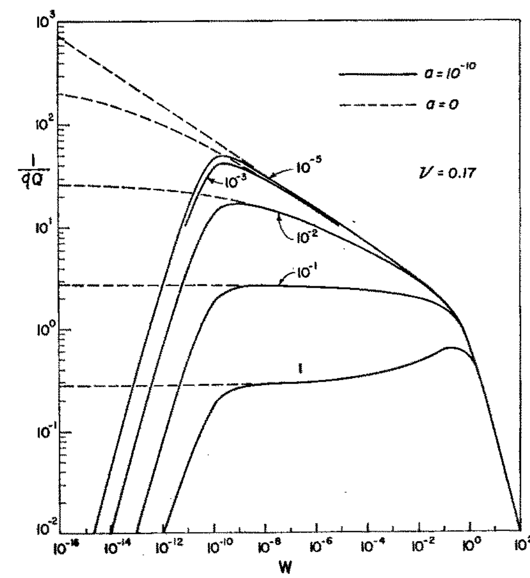

FIG. 2. $1/qQ$ vs normalized frequency W for $\nu=0.17$, $a=10^{-10}$ and zero, and various q values.

transforms. The results of such a procedure are[2,34]

$$\left.\begin{aligned} &J(W)\cong q[\cos W ci|W|-\sin|W| si|W|],\\ &\qquad\qquad\qquad\qquad \nu=0,\\ &H(W)\cong -q\,\mathrm{sgn}W[\sin|W| ci|W|+\cos W si|W|].\end{aligned}\right\} \quad (42)$$

When $W\ll1$, these results become, approximately,

$$\left.\begin{aligned} &J(W)\cong q[\tfrac{1}{2}\pi|W|-\gamma-\ln|W|],\\ &\qquad\qquad\qquad a\ll W\ll 1,\\ &\qquad\qquad\qquad \nu=0,\\ &H(W)\cong q\,\mathrm{sgn}W[\tfrac{1}{2}\pi+|W|\{\gamma+\ln|W|\}].\end{aligned}\right\} \quad (43)$$

$H(W)$ will usually dominate in the expression for $1/Q$ in the present case and will eventually lead to a slow decrease in $1/Q$ as W decreases.[24]

Values of the quantity $1/Qq$ versus W have been calculated from (38) and (39) with a digital computer for various values of a, ν, and q. Some of the results of such calculations are presented in Figs. 1–4. Those figures which do not have a value of a shown on them were computed with $a\ll W$, equivalent to $a=0$. The parameters shown on the curves of Figs. 1–3 are values of q, while those shown on Fig. 4 are ν values. In Fig. 2, the effect of taking $a=10^{-10}$ has been shown and compared with the results obtained with $a\ll W(a=0)$. In general, the curves will begin their left-hand decays for W values near $W=a$, as illustrated in Fig. 2. Since this decaying region is generally far below the low-frequency region accessible to observation, only a single instance of such decay has been shown.

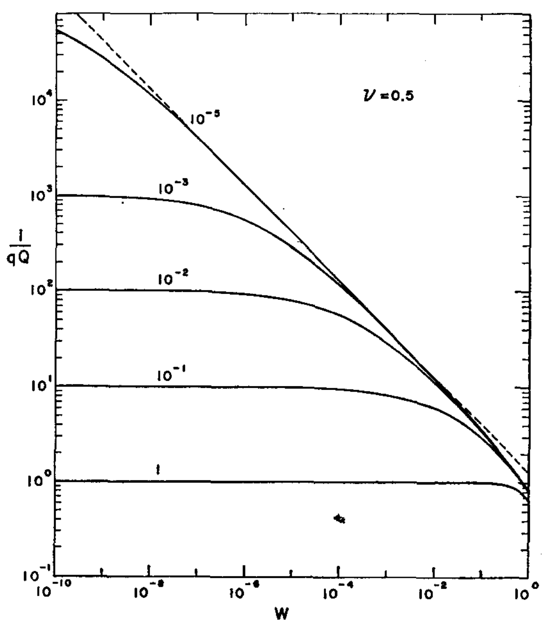

FIG. 3. $1/qQ$ vs normalized frequency W for $\nu=0.5$, $a=0$, and various q values.

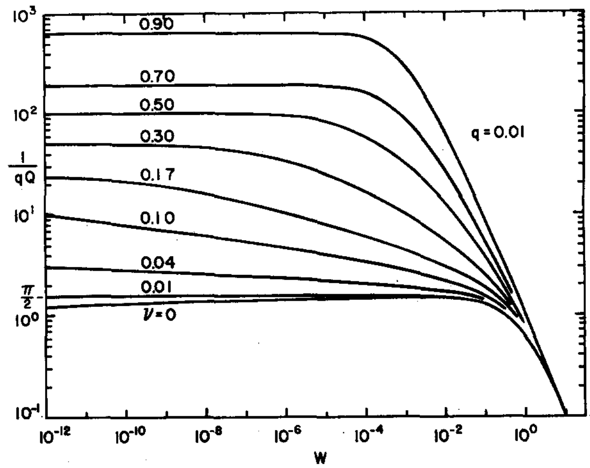

FIG. 4. $1/qQ$ vs normalized frequency W for various ν values, $a=0$, and $q=0.01$.

The level regions where $1/Qq$ is independent of frequency should be especially noted in Figs. 1–4. The values of $1/Qq$ in these regions are consistent with the predictions of Eq. (41) when pertinent. The attenuation and phase factors which follow from (40) when $(1/Q)^2\ll1$ and $\nu>0$ are approximately

$$\left.\begin{aligned} &\alpha\cong q\Gamma(\nu)\sin(\pi\nu/2)W^{1-\nu}/2V_e\tau_0,\\ &\qquad\qquad\qquad 0<\nu<1,\\ &\qquad\qquad\qquad a\ll W\ll1,\\ &\beta\cong W/V_e\tau_0,\end{aligned}\right\} \quad (44)$$

for $J(W)\ll1$, and

$$\left.\begin{aligned} &\alpha\cong[q\Gamma(\nu)\sin(\pi\nu/2)\tan(\pi\nu/2)]^{\frac{1}{2}}W^{1-\nu/2}/2V_e\tau_0,\\ &\qquad\qquad\qquad 0<\nu<1,\\ &\qquad\qquad\qquad a\ll W\ll1,\\ &\beta\cong[q\Gamma(\nu)\cos(\pi\nu/2)]^{\frac{1}{2}}W^{1-\nu/2}/V_e\tau_0,\end{aligned}\right\} \quad (45)$$

for $J(W)\gg1$. Since $\nu\ll1$ is the only condition which results in regions where $1/Q$ is both frequency independent and much less than unity, the usual experimental results, both (44) and (45) show that α will be closely proportional to frequency in the constant $1/Q$ region of interest.

When $a\ll1\ll W$ and $0\leq\nu<1$, expansion of the Lommel-function relations (30′) and (31′) show that for $(1/Q)^2\ll1$,

$$\left.\begin{aligned} &1/Q\cong q/W,\\ &\alpha\cong W/2\tau_0V_eQ=q/2\tau_0V_e, \qquad 0\leq\nu<1,\\ &\beta\cong W/\tau_0V_e. \qquad\qquad a\ll1\ll W,\end{aligned}\right\} \quad (46)$$

In this region where $1/Q$ decreases as W^{-1}, the attenuation factor is frequency independent and determined by the properties q, τ_0, and V_e of the material in question. The frequency dependence of $1/Q$ in the region $W\leq a$

depends somewhat on details of the distribution of relaxation times for $\tau \geq \tau_\infty$. For the present distribution function, $1/Q$ is approximately proportional to W for $W \ll a$ and $(1/Q)^2 \ll 1$.

DISCUSSION

In the following discussion, we shall consider various data to which the present results may apply and shall present a brief summary of work by various authors which is pertinent to the microscopic interpretation of the parameters q, ν, and τ_0. In order to establish firmly the applicability of the present distribution-of-relaxation-times model to the analysis of experimental data, it is necessary to have available either $1/Q$ vs frequency measurements in a range that extends from $W \sim 10$ down to perhaps 10^{-5} or less or to have small-strain creep data in the range of times near τ_0. Data of either type will allow τ_0 to be quite accurately determined if the present results are applicable and should permit reasonable estimates to be made of q and ν as well.

In the usual wave transmission measurements of $1/Q$, the maximum strains produced in the material by a stress wave are of the order of 10^{-5} or less. The specific dissipation factor $1/Q$ is usually found to be independent of strain amplitude in this range for polycrystalline metals[36] and the response of the material may be considered linear. In some metallic single crystals,[37] however, linearity is not reached until the strain amplitude is less than 10^{-7}. On the other hand, most creep measurements involve maximum strains which considerably exceed 10^{-5}. The present theory can, therefore, only be applied to creep experiments in the range of strains for which it has been established that linear response is obtained. As Eq. (2) indicates, this requires that the total strain at a given time after the application of a constant stress σ_0 be proportional to this stress. Furthermore, direct comparison between $1/Q$ results obtained from wave transmission measurements and those obtained for the same material by the application of linear analysis of the present type to creep experiments should only be made if it is certain that the linear range of strains is not exceeded in either type of measurement.

Creep measurements are usually made with either tensile or shear stress applied, while a large number of different kinds of waves such as Rayleigh, shear, and compressional waves may be produced by earthquakes or used for internal friction measurements in solids. Ideally, distinctions should be made between these different phenomena since they will each have specific $1/Q$ functions associated with them.[38] It would obviously be wrong to compare directly $1/Q$ values derived from creep measurements under shear with those obtained from measurements on the same material using compressional waves.

Creep and Internal Friction

Because of the restriction to the linear stress-strain region, viscous or steady-state creep[37] is excluded from present consideration. Even in measurements of transient creep it appears that the linear stress-strain region is often exceeded. For generality, some transient creep results will be mentioned herein, however, for which no test of linearity was made and the linear region may be exceeded. When $(t/\tau_0) \gg 1$ and logarithmic time response of creep is found, Eqs. (6) and (9) may be simplified to show that $(td\epsilon/dt)$ should be a linear function of the applied stress σ_0. Lomnitz[23] has indeed found such linearity for at least the early part of his creep tests on rocks under shear stress. We may thus be assured that the response of the materials was linear over at least part of his total strain range, which extended from about 3×10^{-6} to about 1.5×10^{-4} rad.

Another restriction on creep in the linear region is that it be recoverable or reversible. On removal of the applied stress, the material should creep back toward its original shape, reaching it in the limit of long times. Lomnitz[23] measured such recovery, and it appears likely from his results that most or all of the strain was reversible. Kê[39] has carried out measurements on internal friction and creep in polycrystalline aluminum and finds that the results are mutually consistent, the creep is recoverable, and the material behaves as a viscoelastic solid. He interprets his results on the basis of grain boundary slip and keeps the total shear strain in creep less than 2×10^{-5}, essentially within the range that Mason[36] found the internal friction of polycrystalline aluminum to be amplitude independent.

Two review articles[37,40] and a book[41] have recently appeared which discuss transient creep in metals in some detail. It appears, particularly from the work of Wyatt,[42] that logarithmic creep predominates for polycrystalline metals at relatively low temperatures while Andrade power-law creep,[17] described by $\epsilon(t) \propto t^{\frac{1}{3}}$, yields a better fit to the data at higher temperatures and larger strains. In the present analysis, the basic expression for $\psi(t)$ given in (5) or that in (35) grades almost continuously from the logarithmic form for $\nu=0$ to the Andrade form for $\nu=\frac{1}{3}$. Note, however, that $\nu=\frac{1}{3}$ does not yield exactly the Andrade power law function since the latter predicts an infinite initial rate of creep while (9) and (34) lead to $\sigma_0 q/M\tau_0$ for the initial rate. It is well known that creep increases rapidly as the temperature is raised, and insofar as a generalized power law can describe the creep results on

[36] W. P. Mason, J. Acoust. Soc. Am. **28**, 1197 (1956).
[37] J. Fleeman and G. J. Dienes, *Rheology*, edited by F. R. Eirich (Academic Press, Inc., New York, 1956), Vol. 1, p. 201.
[38] J. R. Macdonald, Geophys. J. **2**, 132 (1959).
[39] T. S. Kê, Phys. Rev. **71**, 533 (1947).
[40] A. H. Sully, *Progress in Metal Physics*, edited by B. Chalmers and R. King (Pergamon Press, New York, 1956), Vol. 6, p. 135.
[41] H. G. Van Bueren, *Imperfections in Crystals* (North-Holland Publishing Company, Amsterdam, 1960).
[42] O. Wyatt, Proc. Phys. Soc. (London) **B66**, 459 (1953).

metals, it appears that ν must be taken temperature dependent, increasing with temperature from a value near or equal to zero which is appropriate at temperatures low compared to the melting point of the material.

The results of Wyatt[42] indicate that in the usual range of temperatures the quantity here denoted $\sigma_0 q/M$ is approximately proportional to absolute temperature. Transient creep still appears at temperatures near absolute zero, however, and the absence of much temperature dependence of q/M in this range indicates that the observed creep is not associated with a thermally activated process.[43,44] Glen[43] has suggested that the actual process involves quantum mechanical tunneling of dislocations through energy barriers which impede their motion and Mott[45] has given this suggestion a theoretical treatment.

Thus far, although it has been shown that experimental creep results for the linear region of strains may be associated with a distribution of relaxation times (actually retardation times), nothing has been said about microscopic processes which might lead to such a distribution. Because of the complexity of the effects observed, only a qualitative discussion is warranted at this time. Since some discussion of this matter for rubbers and polymers has been given,[19,46] the present discussion will be restricted to other materials such as metals and rocks.

Although the results of steady-state internal friction measurements and creep experiments on the same material at the same temperature should be interrelated in the linear range as shown herein and as found experimentally by Kê,[39] separate theories of internal friction and transient creep have been developed in the last few years. This is at least in part because the creep theories are usually intended to apply at higher strains than the theories of internal friction. In the range to which the present results apply, the same time-independent distribution-of-relaxation-times function should be involved in both effects.

All creep theories which have been applied in the temperature range appreciably above absolute zero where tunneling effects are apparently no longer of importance have made use of Becker's[47] idea that thermal fluctuations are necessary to produce flow.[48] The widespread applicability of the logarithmic and power laws of creep to such diverse materials as fibers, polymers, metals, and rocks suggests, as Cottrell[48] has pointed out, that there are general features involved in creep common to all or most solids. A number of thermal activation theories of creep which include the combined action of stress and thermal vibrations have been proposed.[19,28,42,48–53] These theories generally involve energy barriers separating distinct states of the system and transitions between such states of such entities as molecules, molecular groups, flow units, and dislocations or dislocation loops. Several of these creep theories lead to simple or modified logarithmic or power laws.

The dislocation model of internal friction due to Koehler[54] and Granato and Lücke[55,56] treats viscous damping of dislocation lengths or loops which are anchored at pinning points and vibrate under applied stress. Similarly, dislocation theories of creep in metals, associated with the work of Orowan,[49,51] Cottrell,[48] and Mott[50,52] involve the movement and trapping by obstacles of dislocations already present or generated in the material. The obstacles may be other dislocations, foreign atoms, Peierls's hills, nodes of the dislocation network, etc.,[56] and dislocations may be released from them by thermal vibrations and stress. It is clear that even ignoring grain boundary motion effects there are more than sufficient different interactions and processes likely to occur in a real material to account for a distribution of relaxation times. For example, there may be, among others, a distribution of dislocation loop lengths,[54,55] of activation energies or energy levels between different states, of pinning strength, and of different surroundings of different dislocation loops.

In Mott's latest theory,[52] which can lead to logarithmic time dependence, the slowing down of creep with time is ascribed to the increasing difficulty of release of dislocations from obstacles as work hardening progresses. This is a partly irreversible process and so cannot be completely related to the reversible linear processes associated with the very small strains considered in the present work. Consequently, Mott's theory, in common with most of the other thermal-activation theories already cited, does not lead to a linear stress-strain relation. Creep slows down on the present model because there is a lower and lower density of relaxing units which have relaxation times of the order of the time of measurement as this increases beyond $t \sim \tau_0$. Although the linear, reversible character of the present theory requires that the distribution of relaxation times appropriate at a given temperature be a time-independent property of the material, this

[43] J. W. Glen, Phil. Mag. **1**, 400 (1956).

[44] H. M. Rosenberg, *Progress in Metal Physics*, edited by B. Chalmers and R. King (Pergamon Press, New York, 1958), Vol. 7, p. 339.

[45] N. F. Mott, Phil. Mag. **1**, 568 (1956).

[46] Reference 32, pp. 408–413.

[47] R. Becker, Phys. Z. **26**, 919 (1925); Z. tech. Physik **7**, 547 (1926).

[48] A. H. Cottrell, J. Mech. and Phys. Solids **1**, 53 (1952).

[49] E. Orowan, J. West Scot. Iron Steel Inst. **54**, 45 (1946–47).

[50] N. F. Mott and F. R. N. Nabarro, *Report of Conference on Strength of Solids* (Physical Society, London, 1948), p. 1.

[51] E. Orowan, *Imperfections in Nearly Perfect Crystals*, edited by W. Shockley *et al.* (John Wiley & Sons, Inc., New York, 1952), p. 191.

[52] N. F. Mott, Phil. Mag. **44**, 742 (1953).

[53] A. J. Kennedy, J. Mech. and Phys. Solids **1**, 172 (1953).

[54] J. S. Koehler, *Imperfections in Nearly Perfect Crystals*, edited by W. Shockley *et al.* (John Wiley & Sons, Inc., New York, 1952), p. 197.

[55] A. Granato and K. Lücke, J. Appl. Phys. **27**, 583, 789 (1956).

[56] K. Lücke and A. Granato, *Dislocations and Mechanical Properties of Crystals*, edited by J. Fisher *et al.* (John Wiley & Sons, Inc., New York, 1957), p. 425.

does not preclude application of the present results to creep measurements on a previously work-hardened material; it merely restricts the theory to such small deformations that additional work hardening or similar irreversible changes are unimportant during creep or during the transmission of a stress wave.

Both the theories of Mott[52] and of Wyatt[42] lead to direct dependence on absolute temperature of the quantity equivalent to the present $\sigma_0 q/M$, in approximate agreement with Wyatt's experimental results. Note that Eq. (36) shows that q is connected with the value of $\psi(\infty)$, an intrinsic property of the material at a given temperature, since $\psi(\infty)$ is itself dependent on the relaxation time distribution.

Although Mott's work-hardening theory is not closely applicable to the present model, it is worth mentioning that it yields an analytic expression for a time constant such as the τ_0 which appears, for example, in (4). Mott[52] has shown that reasonable values for the microscopic parameters determining τ_0 lead to a value for it in order of magnitude agreement with the figure of 1 sec found experimentally by Davis and Thompson[57] for a precipitation-hardened Cu-Ag alloy at room temperature. As we shall see in the next part, the figure of $\tau_0 \sim 1$ sec is also in fair agreement with the τ_0 obtained by fitting the present theory to $1/Q$ results associated with long-period phenomena.

The magnitude of τ_0 is a critical factor in the present theory because it determines the frequency range where $1/Q$ changes from being roughly constant to decreasing as ω^{-1}. It also determines the time at which the creep rate first deviates from its initial value and becomes time dependent. Unfortunately, most creep measurements have not been carried out to sufficiently short times or analyzed properly to give good values of τ_0. Lomnitz's measurements for creep in rock[23] start at about 30 sec after the application of shear stress, yet he derives a value of τ_0 (his a^{-1}) of about 10^{-3} sec. Since his results are well fitted in the range of measurement by simple logarithmic time dependence, a value of τ_0 far less than the shortest time of measurement cannot be extracted from them. It can only be concluded from his work that τ_0 for rocks is less than 30 sec. Wyatt's creep measurements on pure metals[42] begin at about 2 sec and allow us to infer that τ_0 is likely to be less than this for such materials.

Whenever creep measurements for $t \ll \tau_\infty$ can be fitted by a form like Eq. (2) with the expression (5) used for $\psi(t)$, τ_0 can probably be obtained most accurately by analyzing $d\epsilon/dt$ results. For $t>0$ we may write

$$d\epsilon/dt = (\sigma_0 q/M\tau_0)[1+(t/\tau_0)]^{\nu-1}. \quad (47)$$

Let us denote by B the quantity $(\sigma_0 q/M\tau_0)^{1/(\nu-1)}$. Then (47) may be rewritten as

$$(d\epsilon/dt)^{1/(\nu-1)} = B + (B/\tau_0)t. \quad (47')$$

If now the quantity on the left is plotted versus t for various values of ν near and including zero, one such value of ν should lead to the best straight line, from which τ_0, ν, and B can be obtained. A somewhat similar procedure was used by Davis and Thompson[57] to obtain the value of τ_0 already quoted, although they did not fit their data to a modified logarithmic form like that in Eq. (4). Further discussion will be given later of the quantities ν, q, and τ_0 which appear in the present work.

The time constant τ_∞ cannot be determined directly unless the data can be extended to the region $W \leq a$ or $t \geq \tau_\infty$. We have found no data pertaining to sufficiently long times or low frequencies that these conditions are reached. In addition, without data in this region to allow more precise definition of the form of the relaxation spectrum at these long relaxation times, the present specific modification of the distribution-of-relaxation-times function is somewhat speculative. Nevertheless, some such modification is necessary to make the theory describe a physically realizable system. If the present distribution of relaxation times is accepted in lieu of better information, τ_∞ is well defined theoretically and an estimate of its magnitude can be obtained from the ratio of final to initial strain predicted by the present theory when all the parameters involved are well known. Jeffreys[26] has somewhat arbitrarily taken a value of 1.1 for this ratio for the earth and states that there is good evidence that it is less than 1.6. The present results yield

$$\epsilon(\infty)/\epsilon(0) = 1 + \psi(\infty), \quad (48)$$

where $\psi(\infty)$ is given by (36) or (37). If we take the approximate values $\nu=0$, $q=10^{-2}$, and $\tau_0=8.5$ sec, which are found in the next part, and use 1.1 for the above ratio, τ_∞ is about 1.4 days, manifestly too short. However, for the value $q=10^{-2}$ used here, τ_∞ increases by e^{10} for each 0.1 increase in $\epsilon(\infty)/\epsilon(0)$ when $\nu=0$. Therefore, if this ratio is taken to be 1.4, τ_∞ is about 4×10^{10} yr, a sufficiently long time. The quantity τ_∞ decreases as ν increases. For $\epsilon(\infty)/\epsilon(0)=1.1$, one finds $\tau_\infty \sim 0.3$ day for $\nu=0.01$ and 2.1 hr for $\nu=0.04$. When the ratio is 1.4, $\tau_\infty \sim 2.3\times 10^7$ yr and 1.3×10^3 yr for $\nu=0.01$ and 0.04, respectively. The $a^{-\nu}e^{-a}$ term in (36), which arises from taking $G(\tau)>0$ for $0<z\leq a$, has only a relatively small effect on the above values. Because of the strong dependence of τ_∞ on the uncertain quantities $\epsilon(\infty)/\epsilon(0)$, q, and ν, we can only conclude that within their present limits of uncertainty they can indeed lead to a large enough τ_∞ value to have escaped direct observation.

Long-Period Phenomena

Transient creep is a time-domain phenomenon concerned with strain and its rate of change. We shall now pass on to small-amplitude frequency-domain measurements associated with vibration, oscillation,

[57] M. Davis and N. Thompson, Proc. Phys. Soc. (London) **B63**, 847 (1950).

and wave motion. Disagreement between results obtained by transient and frequency response methods on the same material may indicate that the range of linearity has been exceeded. Such a conclusion is only justified, however, provided that the time span in the transient measurements is consonant with the range of periods in the frequency response measurements. The geophysical results discussed in this section are roughly consonant in this way only with usual creep measurements.

In Fig. 5, $1/Q$ values primarily pertaining to wave phenomena in the earth are plotted vs angular frequency. Data for body waves have been indicated with an open symbol while a solid symbol or line has been used for data pertaining to surface and mantle waves. When Q is large, Eq. (20) shows that $1/Q$ and the attenuation factor α are related by

$$1/Q = \alpha t_0 V/\pi, \tag{49}$$

where t_0 is the period and V the phase velocity of the waves. The logarithmic decrement δ is given approximately by π/Q and the time lag, or time of retardation, by $1/\omega Q$. The data shown on Fig. 5 have been obtained from a variety of sources,[58–69] and it is surprising that they are as consistent as indicated. For convenience, the type of phenomenon considered has been listed after each of references 58 through 69. For the data shown as horizontal lines, $1/Q$ was roughly constant over the limits indicated. The S-wave point which includes an arrow represents a lower limit. The extremes of the vertical lines indicate points obtained from measurements associated with two different earthquakes.

In the region where $1/Q$ decreases as ω^{-1}, Eq. (49) indicates that αV is constant. Since Q is high in this region, dispersion of the velocity will be negligible and α will be essentially frequency independent, in agreement with Eq. (46). There is sufficient data on Fig. 5 to allow theoretical curves to be fitted approximately. The results of such fitting are indicated by the solid lines. The values used for the theoretical lines were $q = 10^{-2}$ and $\tau_0 = 8.5$ sec. Since the beginning of a bend

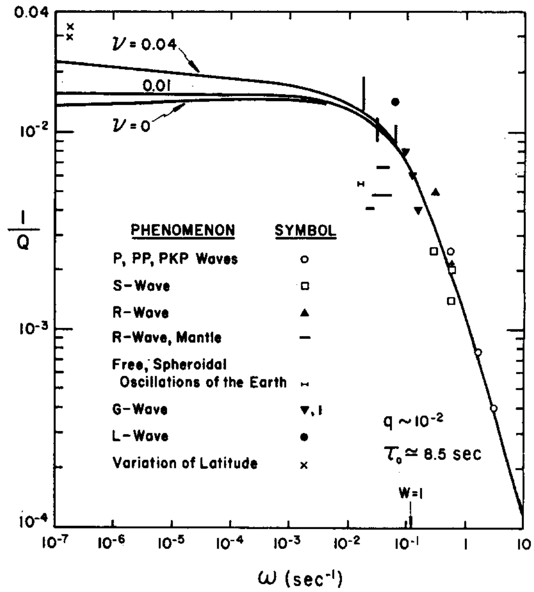

FIG. 5. $1/Q$ vs angular frequency ω for $\nu = 0$, 0.01, and 0.04; $a = 0$; $\tau_0 = 8.5$ sec; and $q = 0.01$. A variety of experimental points are also shown.

can be observed in the data for longer periods, the estimate for τ_0 is probably accurate to within 50% or less.

The absence of much data at very long periods makes the determination of q, and even more so ν, very uncertain. A damping value for the semi-diurnal earth tides has not been plotted because of scatter arising from the effects of ocean loading. Data obtained from mid-continental stations suggests[70] that the appropriate $1/Q$ with such loading minimized may be as low as 5×10^{-3}. The two values of $1/Q$ obtained from the 14-month variation of latitude of the earth are quite uncertain[68,70] and represent over-all system values which include interface dissipation, especially that at the boundary between the core and the mantle and that between the mantle and the oceans.[70] Without these dissipations, the over-all value of $1/Q$ might be 5×10^{-3} or smaller. A small negative value of ν would lead to a curve falling below that for $\nu = 0$ in Fig. 5 and might be more appropriate in this case.

It will be noted from Fig. 5 that the $1/Q$ values for the body waves which travel in both core and mantle are consistently lower than those that are restricted to the mantle and surface alone. At least part of such a difference might arise from the higher Q to be expected for material under high pressure in the core. In addition, the uncertain and varying effects of scattering arising from inhomogeneities along the paths of the various waves involved in Fig. 5 have not been estimated and corrected for. Further, no account has been taken of the fact that the $1/Q$ values for Rayleigh, shear, and compressional waves of the same period traveling in the same medium are in general different but related.[38]

[58] B. Gutenberg, Bull. Seismol. Soc. Am. **48**, 269 (1958). *P*, *PP*, *PKP*, *S* waves.

[59] F. Press, Science **124**, 1204 (1956). *S* waves.

[60] B. Gutenberg, Bull. Seismol, Soc. Am. **35**, 3 (1945). Rayleigh waves.

[61] M. Ewing and F. Press, Bull. Seismol. Soc. Am. **44**, 127 (1954). Mantle Rayleigh waves.

[62] M. Ewing and F. Press, Bull. Seismol. Soc. Am. **44**, 471 (1954). Mantle Rayleigh waves.

[63] H. Benioff, F. Press, and S. Smith, J. Geophys. Research **66**, 605 (1961). Free, spheroidal oscillations of the earth.

[64] M. Bath, Geofis. pura e appl. **41**, 91 (1958). Mantle Rayleigh waves.

[65] Y. Satô, Bull. Seismol. Soc. Am. **48**, 231 (1958). Gutenberg waves (mantle Love waves).

[66] B. Gutenberg, Phys. Z. **25**, 377 (1924). Love waves.

[67] H. Jeffreys, *The Earth* (Cambridge University Press, Cambridge, England, 1959), 4th ed., p. 255.

[68] W. H. Munk and G. J. F. MacDonald, *The Rotation of the Earth* (Cambridge University Press, Cambridge, England, 1960). Variation of latitude.

[69] P. Fellgett, work in progress: quoted in reference 68. Variation of latitude.

[70] G. J. F. MacDonald (private communication).

Here, the situation is complicated by unequal periods for the waves and, in view of the uncertainty of the data and even uncertainty in the applicability of the present theory to the raw data presented, it has not been felt worthwhile to try to take the above effect into account.

It is a hazardous extrapolation to expect the present theory to describe loss in the core, mantle, and crust simultaneously in a very accurate fashion.[71] Therefore, the theoretical fitting shown in Fig. 5 can only be expected to give order of magnitude values. It is suggestive, however, that the present τ_0 value is reasonably consistent with the results of creep measurements on metals and rocks already mentioned. If sufficient accurate wave transmission data for the earth were available, especially at long periods, it might be preferable to analyze it into two (or more) dispersion regions, a high-Q one pertaining to the average core, and a lower-Q, perhaps half as large, for the average mantle. The present theoretical results could be applied separately to these regions with different values of τ_0, q, and possibly ν for each. An over-all response could then be obtained by addition of $1/Q$ values for the separate regions.

The present rough curve fitting and previously considered creep data both suggest that the most likely value of ν for rocks and metals in the small strain range is near zero and is almost certainly less than 0.05. Jeffreys[27] has derived an approximate value of $\nu=0.17$ using (5) and assuming, on the basis of Lomnitz's erroneous value $\tau_0^{-1}=10^3$, that $W\ll 1$ for seismic waves. The data and curve fitting of Fig. 5 show that this assumption is unwarranted. Furthermore, Fig. 2 indicates that the combination of a sufficiently high Q to agree with experiment and an appreciable region of frequency independence of $1/Q$ is unrealizable for ν as large as 0.17.

Short-Period Phenomena

Acoustic, seismological, and vibrational determinations of energy loss in solids are generally carried out under conditions which involve very small strains. Therefore, the present linear theory should be particularly applicable to the analysis of such amplitude-independent measurements. A variety of measurements have recently been summarized by Knopoff and MacDonald[72] who conclude that $1/Q$ for inorganic, nonferromagnetic materials is very nearly frequency-independent over the range from 10^{-2} to 10^7 cps, a conclusion in some apparent disagreement at the lower end with the data of Fig. 5. No single material has been measured at all frequencies of this entire range, however. The magnitude of $1/Q$ usually found is of the order of 10^{-2} to 10^{-3}, although it may be even smaller. Other internal friction data discussed in an article by Niblett and Wilks,[73] which also includes a discussion of mechanisms leading to internal friction, show some frequency dependence of $1/Q$ (or logarithmic decrement) over small frequency ranges, but the majority of data leads to very weak or absent frequency dependence over a wide frequency range. Kolsky[74] has presented data on several non-metallic materials which also show negligible frequency dependence.

Measurements on a number of solids over a limited frequency range indicate that the attenuation factor is rather closely proportional to frequency. Equation (49) cannot be used to establish the frequency independence of $1/Q$ from such results unless the phase velocity is shown to be frequency-independent. Equations (44) and (45) of the present theory indicate that for $a\ll W\ll 1$ and $0<\nu\ll 1$, the frequency dependence of α will be very difficult to distinguish experimentally from direct proportionality. Alternatively, when $\nu=0$ and Q is large, Eqs. (17) and (43) indicate that α is given quite closely for $a\ll W\ll 1$ by

$$\alpha\cong(W/2\tau_0V_e)H(W)[1+J(W)]^{-\frac{1}{2}}. \qquad (50)$$

Here, α may increase slightly more rapidly than direct frequency proportionality over an appreciable range, but the deviation will be difficult to detect experimentally.

Of especial interest are stress attenuation results obtained on granite and other surface rocks. The measurements of Bruckshaw and Mahanta[75] indicate that $1/Q$ is frequency independent for several such materials from 40–120 cps. Others[72,76–78] have found evidence of such frequency independence up to 10 Mcps and down to lower seismic frequencies. These results seem difficult to reconcile with those of Fig. 5 at first. It should be noted, however, that except for recent results on Solenhofen limestone[79] there are no accurate $1/Q$ data available for rocks in the frequency region from about 1 cps to 20 or 30 cps. The above discrepancy may be reconciled in one of two ways. First, the decrease in $1/Q$ shown at the right in Fig. 5 may not arise primarily from the present loss mechanism but may appear because of the increase in Q with increasing pressure and even from a possible decrease in scattering effects at shorter wavelengths. Thus, the short-period body waves which penetrate deeper into

[71] B. Gutenberg, *Physics of the Earth's Interior* (Academic Press, Inc., New York, 1959), p. 191.
[72] L. Knopoff and G. J. F. MacDonald, Revs. Modern Phys. **30**, 1178 (1958).
[73] D. H. Niblett and J. Wilks, Advances in Phys. **9**, 1 (1960).
[74] H. Kolsky, *Stress Wave Propagation in Materials*, edited by N. Davids (Interscience Publishers, Inc., New York, 1960), p. 59.
[75] J. Mc. Bruckshaw and P. C. Mahanta, Petroleum **17**, 14 (1954).
[76] F. Birch and D. Bancroft, Bull. Seismol. Soc. Am. **28**, 243 (1938).
[77] Ye. V. Karus and I. P. Passechnik, Akad. Nauk S.S.S.R., Izvest. Ser. Geofiz. **6**, 514 (1954).
[78] L. Peselnick and I. Zietz, Geophysics **24**, 285 (1959).
[79] L. Peselnick and W. F. Outerbridge, J. Geophys. Research **66**, 581 (1961).

the earth would have most of their path lengths in high-Q core material. In this case, the hypothesis of frequency-independence of $1/Q$ for many materials at constant pressure and temperature over the range from 10^{-2} or 10^{-1} to 10^7 cps may be accepted, although even so the lower end of the range seems somewhat difficult to reconcile with creep results.

Secondly, if pressure and scattering effects are actually unimportant and the present model is assumed to apply to the lower frequencies such as those covered by Fig. 5 with $\tau_0 \sim 10$ sec, new loss mechanisms will begin to be important at higher frequencies. Somewhere in the region of 1 to 20 cps, the $1/Q$ decrease shown in Fig. 5 would then cease and either $1/Q$ would remain constant at a new lower value or it would, more likely, increase up to 10^{-3} to 10^{-2} again and then remain constant to very high frequencies. In either case, this new dispersion region might again be well described by a linear loss mechanism of the present type with values of τ_∞ and τ_0 different from those applicable to the lower dispersion region, and values of q and ν either the same or different. It is known that $1/Q$ is considerably less in single crystals than in polycrystalline aggregates of the same material. It is possible that the higher-frequency dispersion region begins to become important in polycrystalline material when the wavelength is short enough that the boundaries between individual grains and crystallites can begin to play an important role in energy dissipation.[39] Under these conditions, one would naturally expect a distribution of relaxation times. At the longer wavelengths of Fig. 5, grain-boundary effects and degree of compaction might be expected to be less important. Knopoff and MacDonald[4] have considered various models for acoustic loss in solids and conclude that microscopic scattering effects will not become important until very high frequencies are reached. The very recent results of Peselnick and Outerbridge[79] on internal friction in Solenhofen limestone indicate a nearly constant $1/Q$ from about 4 cps up to the limit of acoustic frequencies or above. For this material at least it is therefore likely that there is only one dispersion region rather than two separate ones with a transition region in the 1–10 cps frequency range.

If the hypothesis of separate upper and lower dispersion regions is accepted, the present analysis can also explain the experimental constancy of $1/Q$ to very high frequencies and the nearly direct proportionality of α to frequency. In the region from perhaps $\omega = 10^2$ to 10^8, $1/Q$ remains at least roughly constant and of the order of 10^{-3} to 10^{-2}. When $\nu = 0$, Eqs. (19) and (43) show that $1/Q \sim \pi q/2$, thus setting limits on q. Alternatively, if $0 < \nu < 1$, Eq. (41) requires $\nu \ll 1$ for essentially frequency independent $1/Q$ coupled with a sufficiently large value of Q. If $W = a$ at $\omega \cong 10^2$, the τ_∞ for this dispersion region will be 10^{-2} sec. Further, if $W = 1$ at $\omega \cong 10^8$, τ_0 will be 10^{-8} sec. These results are predicated on sufficient separation of the low- and high-frequency dispersion regions that they can be represented by two essentially nonoverlapping distributions of relaxation times. If overlapping is considerable, a single relaxation distribution function must be used which covers both dispersion regions, and the distinction between the τ_0 for the lower region and the τ_∞ for the upper region will become blurred or disappear. If the decrease in $1/Q$ shown in Fig. 5 is largely a pressure or boundary-layer artifact associated with mantle and core wave paths, the distinction disappears completely and small amplitude loss in a given material such as a metal or rock should be describable by the present theory using a τ_∞ between 10^2 and 10^9 yr and a τ_0 of 10^{-8} sec or less, a remarkably wide range of times and wide distribution of relaxation times.

The usual theory of amplitude-independent internal friction losses [54-56] leads to an expression for $1/Q$ which is proportional to the mean length of a vibrating dislocation loop raised to the fourth power and directly proportional to frequency for frequencies appreciably below 100–1000 Mcps. As the foregoing discussion indicates, such frequency dependence is in poor agreement with most experiments. Wilks[80] has recently suggested, however, that the L^4 term may be an inverse function of frequency. Such dependence of L will make the over-all frequency dependence of $1/Q$ less strong and may even lead to a decrease in internal friction at high frequencies. Wilks' argument is that the effective length of a loop will decrease with frequency for frequencies appreciably greater than the inverse of the average time required for the unpinning of a dislocation from an impurity atom. Although Wilks relates this time to an activation energy which may depend on the nature of the impurity and the distance to adjacent pinning points, he does not explicitly consider a distribution of times and activation energies. A distribution of activation energies is certainly likely and can lead to a distribution of relaxation times and, equivalently, a dependence of effective loop length on frequency.

The problem of explaining a dissipation factor which is nearly independent of frequency over many decades also arises in the field of dielectrics. A number of authors have attacked the dielectric problem by also considering a distribution of activation energies.[81–83] For example, one may often write

$$\tau = \tau_a \exp(E/kT), \tag{51}$$

where E is an effective activation energy for the process and may be discontinuously or continuously distributed over a finite range from E_1 to E_2.

If $K(E)dE$ is the density of relaxation times having activation energies in the range dE, then $K(E)dE = G_1(\tau)d\tau$. Since $K(E)$ should be a temperature-

[80] J. Wilks, Phil. Mag. 4, 1379 (1959).
[81] M. Gevers and F. K. Du Pré, Trans. Faraday Soc. **A42**, 47 (1946).
[82] C. G. Garton, Trans. Faraday Soc. **A42**, 56 (1956).
[83] H. Fröhlich, *Theory of Dielectrics* (Clarendon Press, Oxford, England, 1958), 2nd ed., pp. 92–98.

independent, time-invariant property of the system, $G_1(\tau)(d\tau/dE)$ should also be temperature independent.[84] This requirement imposes restrictions on the form of a distribution-of-relaxation-times function which can be used to describe an activated process. In the present case when $G(z)$ is given by Eqs. (28) or (33), these restrictions cannot be met exactly, primarily because of the presence of the e^{-z} term. Note that the $G(z)$ of Eq. (33) is still physically realizable, however, when the distribution of relaxation times considered does not arise solely from a distribution of activation energies. The actual distribution function of Eq. (28) multiplied by z^{-1} is, in fact, of the form of a Poisson distribution, a distribution commonly met in nature. For example, such a distribution (with $\nu<0$ and integral) has been found to describe the size statistics of polycrystalline aggregates under some conditions.[85]

If the activation energy range extends from E_1 to E_2, the smallest and largest relaxation times are $\tau_s = \tau_a \exp(E_1/kT)$ and $\tau_l = \tau_a \exp(E_2/kT)$. If τ_l is identified with the present τ_∞, then the minimum value of z is a and $G(z)=0$ for $z<a$, a possibility already mentioned. The quantity τ_s may be set equal to τ_0 without appreciably changing the transient or frequency response results already considered in the range of primary interest. Then the maximum value of z is unity. One will still find a wide frequency region where $1/Q$ will be constant and $1/Q$ will still decrease as W^{-1} for $W\gg1$. Note that even if the intrinsic vibration time τ_a is very small, the time $\tau_s=\tau_0$ may be made a second or longer by taking E_1, the minimum height of a potential barrier, sufficiently large. If E_1 is taken zero or proportional to temperature, τ_0 will be temperature independent.

When τ is given by Eq. (51) and $G(z)$ is expressed by (28) in the range $a\leq z\leq 1$ and is zero outside this range, it turns out that the resulting $K(E)$ may be made very nearly temperature independent when $\nu\ll1$ by taking ν and q both proportional to absolute temperature, results in at least qualitative agreement with experiment. The temperature dependence of the e^{-z} term which appears in $K(E)$ cannot be removed by the above choices but such dependence will be quite small for the present choice of $\tau_0=\tau_s$. Distribution functions which are entirely consistent with thermally activated processes having a distribution of activation energies and which can be related to both dielectric and internal friction behavior will be discussed elsewhere.[84]

The present theory is linear in that superposition applies, and it involves a linear dependence of strain on stress at a given time even though the creep function itself may be a nonlinear function of time, such as that in Eq. (5). Knopoff and MacDonald[72] have stated, however, that no model of dissipation in solids which is based on a linear stress-strain relation can account for the frequency independence of $1/Q$ observed for many materials. This conclusion is based on an approximate treatment of a lumped-parameter model of a linear solid. It has led them, as already mentioned, to propose a nonlinear theory which leads to such constancy of $1/Q$ in the range where $1/Q$ is amplitude independent. The present work, based on a continuously distributed model, shows that a linear theory can also lead to a $1/Q$ which is virtually frequency independent over a wide range of frequencies.

Two of the particular virtues of the present approach are that Q increases at each end of the constant $1/Q$ range. Such increase causes the effects of a given dispersion mechanism to be of importance in a limited, but possibly very wide, frequency range only. Outside this range, other dispersion mechanisms may come into play which themselves may possibly be describable by a theory of the present type with different material constants. Most other theories[55,72,74] lead to Q's which decrease indefinitely at high frequencies and often at low frequencies as well, precluding the simple superposition of similar solutions to cover several different dispersion regions in the same material.

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[84] J. R. Macdonald (submitted to J. Chem. Phys.).
[85] P. J. Gellings, Appl. Sci. Research **A10**, 165 (1961).